PART E

First order examples and applications

CHAPTER 21

Fuzzy relations

In this chapter three general families of examples (i.e., models) of IOS are introduced, from which a number of more particular ones are obtained in the next three chapters. The initial IOS-operations are explicitly characterized. A $\langle\ \ \rangle$-elimination theorem is used to show that translation in these examples can be implemented by means of an extra counter. Certain specific t-operations are also studied.

Assume that M is a nonempty set, f_1, f_2 is a splitting scheme for it, E is a complete lattice with at least two distinct members and the distributive law

$$\inf\{a, \sup E_1\} = \sup\{\inf\{a,b\}/b\in E_1\}$$

holds for all $a\in E$, $E_1 \subseteq E$. (A *complete lattice* is a partially ordered set all of whose subsets have least upper bounds; the existence of greatest lower bounds follows, for $\inf E_1 = \sup\{a\backslash \forall b\in E_1(a\leq b)\}$.) Writing $\perp$, $\top$ respectively for $\sup\varnothing$, $\sup E$, we have $\perp = \inf E$, $\top = \inf\varnothing$ and $\perp < \top$.

The pairing space of 16.4 is the starting point for the following example which corresponds to example 17 in Skordev [1980], chapter 3. If the reader is not interested in fuzzy relations, then she or he may assume $E = \{\perp, \top\}$ and interpret $\varphi(s,t) = \top$, $\varphi(s,t) = \perp$ respectively as $(s,t)\in\varphi$, $(s,t)\notin\varphi$ (or vice versa), so that the semigroup $\mathscr{F}$ below will then consist of ordinary relations.

Proposition 21.1 (Example 21.1). Take $\mathscr{F} = \{\varphi/\varphi: M^2 \to E\}$ (all the binary E-valued *fuzzy relations* over M), $\varphi \leq \psi$ iff $\forall st(\varphi(s,t) \leq \psi(s,t))$, $I(s,s) = \top$ and $I(s,t) = \perp$ otherwise, $\varphi\psi(s,t) = \sup_r \inf\{\varphi(s,r), \psi(r,t)\}$, $(\varphi,\psi)(f_1(s),t) = \varphi(s,t)$, $(\varphi,\psi)(f_2(s),t) = \psi(s,t)$ and $(\varphi,\psi)(s,t) = \perp$ otherwise, $L = \lambda st.I(f_1(s),t)$ and $R = \lambda st.I(f_2(s),t)$. Then $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is a $(**)_0$, $(***)_0$-complete OS.

Proof. We have

$$\begin{aligned}
\varphi(\psi\chi)(s,t) &= \sup_r \inf\{\varphi(s,r), \psi\chi(r,t)\} \\
&= \sup_r \inf\{\varphi(s,r), \sup_u \inf\{\psi(r,u), \chi(u,t)\}\} \\
&= \sup_r \sup_u \inf\{\varphi(s,r), \psi(r,u), \chi(u,t)\} \\
&= \sup_u \sup_r \inf\{\varphi(s,r), \psi(r,u), \chi(u,t)\}
\end{aligned}$$

$$= \sup_u \inf \{ \sup_r \inf \{ \varphi(s,r), \psi(r,u) \}, \chi(u,t) \}$$

$$= \sup_u \inf \{ \varphi\psi(s,u), \chi(u,t) \} = (\varphi\psi)\chi(s,t)$$

for all s,t, hence $\varphi(\psi\chi) = (\varphi\psi)\chi$. We also get $\varphi I = \lambda st.\sup_r \inf\{\varphi(s,r), I(r,t)\} = \varphi$ and similarly $I\varphi = \varphi$. Therefore, $\mathscr{F}$ is a semigroup with unit I.

All the subsets of $\mathscr{F}$ have least upper bounds, namely

$$\sup \mathscr{H} = \lambda st.\sup \{ \theta(s,t)/\theta \in \mathscr{H} \}.$$

Whenever $\mathscr{H} \subseteq \mathscr{F}$, $\varphi = \sup \mathscr{H}$ and $\psi \in \mathscr{F}$, then

$$\begin{aligned}
\varphi\psi(s,t) &= \sup_r \inf \{ \varphi(s,r), \psi(r,t) \} \\
&= \sup_r \inf \{ \sup \{ \theta(s,r)/\theta \in \mathscr{H} \}, \psi(r,t) \} \\
&= \sup_r \sup \{ \inf \{ \theta(s,r), \psi(r,t) \} / \theta \in \mathscr{H} \} \\
&= \sup \{ \sup_r \inf \{ \theta(s,r), \psi(r,t) \} / \theta \in \mathscr{H} \} \\
&= \sup \{ \theta\psi(s,t)/\theta \in \mathscr{H} \}
\end{aligned}$$

for all s,t; hence $\varphi\psi = \sup(\mathscr{H}\psi)$, while the equalities

$$\begin{aligned}
\psi\varphi(s,t) &= \sup_r \inf \{ \psi(s,r), \varphi(r,t) \} \\
&= \sup_r \inf \{ \psi(s,r), \sup \{ \theta(r,t)/\theta \in \mathscr{H} \} \} \\
&= \sup_r \sup \{ \inf \{ \psi(s,r), \theta(r,t) \} / \theta \in \mathscr{H} \} \\
&= \sup \{ \psi\theta(s,t)/\theta \in \mathscr{H} \}
\end{aligned}$$

imply $\psi\varphi = \sup \psi\mathscr{H}$. In particular, multiplication is monotonic.

Take $L_1 = \lambda st.I(f_1(t), s)$ and $R_1 = \lambda st.I(f_2(t), s)$. It follows that $LL_1 = RR_1 = I$ and $LR_1 = RL_1 = O = \lambda st.\bot$ using the fact that f_1, f_2 is a splitting scheme for M. We get $(\varphi, \psi) = \sup\{L_1\varphi, R_1\psi\}$; hence $\mathscr{S}$ is a $(**)_0$, $(***)_0$-complete OS by 19.4. The proof is complete.

The above IOS can be obtained in another way. Namely, take the pairing space of 16.4 with $N = M$ and augment it with multiplication, then use 16.10, 19.1–19.3.

In order to further sharpen the characterizations given in 19.6, consider the sets $f_2^n(f_1(M))$, $n \geq 0$. It follows that whenever $i \neq j$, then $f_2^i(f_1(M)) \cap f_2^j(f_1(M)) = \varnothing$. Therefore, for every $s \in M$ there is at most one n (if any) such that $s \in f_2^n(f_1(M))$.

Proposition 21.2. Let $\mathscr{S}$ be the IOS of example 21.1 or a subspace of it. Then

$$\langle \varphi \rangle (f_2^n(f_1(s)), f_2^n(f_1(t))) = \varphi(s,t)$$

and $\langle\varphi\rangle(s,t)=\perp$ otherwise,

$$\Delta(\varphi,\psi)(f_2^n(f_1(s)),t)=\varphi\psi^n(s,t)$$

and $\Delta(\varphi,\psi)(s,t)=\perp$ otherwise,

$$[\varphi](s,t)=\sup\{\inf_{i<n}\varphi(f_2^{-1}(r_i),r_{i+1})/n\in\omega\,\&\,r_0=s$$
$$\&\,r_0,\ldots,r_{n-1}\in f_2(M)\,\&\,r_n=f_1(t)\}.$$

Proof. We have $R_1\varphi(f_2(s),t)=\varphi(s,t)$ and $R_1\varphi(s,t)=\perp$ otherwise, while $L_1\varphi(f_1(s),t)=\varphi(s,t)$ and $L_1\varphi(s,t)=\perp$ otherwise. Proposition 19.6 gives $\Delta(\varphi,\psi)=\sup_n R_1^n L_1\varphi\psi^n$, hence $\Delta(\varphi,\psi)(f_2^n(f_1(s)),t)=\varphi\psi^n(s,t)$ and $\Delta(\varphi,\psi)(s,t)=\perp$ otherwise. In particular, $\langle\varphi\rangle(f_2^n(f_1(s)),t)=\varphi LR^n(s,t)$ and $\langle\varphi\rangle(s,t)=\perp$ otherwise. It also follows that $\varphi LR^n(s,f_2^n(f_1(t)))=\varphi(s,t)$ and $\varphi LR^n(s,t)=\perp$ otherwise, hence $\langle\varphi\rangle(f_2^n(f_1(s)),f_2^n(f_1(t)))=\varphi(s,t)$ and $\langle\varphi\rangle(s,t)=\perp$ otherwise.

Proposition 19.6 gives $[\varphi]=\sup_n(R_1\varphi)^n L_1$, hence $[\varphi](s,t)=\sup_n(R_1\varphi)^n L_1(s,t)$ for all s,t. As already mentioned, $R_1\varphi(f_2(s),t)=\varphi(s,t)$ and $R_1\varphi(s,t)=\perp$ otherwise. Using this and the equality $\varphi L_1(s,t)=\varphi(s,f_1(t))$, we get the desired characterization of [ ]. The proof is complete.

The following example corresponds to the pairing space of 16.4 again, this time with $N\times M$ playing the role of N.

Proposition 21.3 (Example 21.2). Let $N\neq\emptyset$. Take $\mathscr{F}=\{\varphi/\varphi:M\times N\times M\to E\}$, $\varphi\leq\psi$ iff $\varphi(s,x,t)\leq\psi(s,x,t)$ for all $s,t\in M$ and $x\in N$, $\varphi\psi(s,x,t)=\sup_r\inf\{\varphi(s,x,r),\psi(r,x,t)\}$, $(\varphi,\psi)(f_1(s),x,t)=\varphi(s,x,t)$, $(\varphi,\psi)(f_2(s),x,t)=\psi(s,x,t)$ and $(\varphi,\psi)(s,x,t)=\perp$ otherwise, $I(s,x,s)=\top$ and $I(s,x,t)=\perp$ otherwise, $L=\lambda sxt.I(f_1(s),x,t)$ and $R=\lambda sxt.I(f_2(s),x,t)$. Then $\mathscr{S}=(\mathscr{F},I,\Pi,L,R)$ is a $(**)_0$, $(***)_0$-complete OS.

The proof repeats that of 21.1 with members of N treated as parameters.

Proposition 21.4. Let $\mathscr{S}$ be the IOS of example 21.2 or a subspace of it. Then

$$\langle\varphi\rangle(f_2^n(f_1(s)),x,f_2^n(f_1(t)))=\varphi(s,x,t),\Delta(\varphi,\psi)(f_2^n(f_1(s)),x,t)=\varphi\psi^n(s,x,t)$$

and $\langle\varphi\rangle(s,x,t)$, $\Delta(\varphi,\psi)(s,x,t)=\perp$ otherwise,

$$[\varphi](s,x,t)=\sup\{\inf_{i<n}\varphi(f_2^{-1}(r_i),x,r_{i+1})/n\in\omega\,\&\,r_0=s$$
$$\&\,r_0,\ldots,r_{n-1}\in f_2(M)\,\&\,r_n=f_1(t)\}.$$

The proof repeats that of 21.2.

The following two statements establish a close two-way connection between examples 21.1, 21.2.

Proposition 21.5. Let $\mathscr{S}_1,\mathscr{S}_2$ be the IOS respectively of examples 21.1, 21.2 based on the same sets M, splitting schemes and lattices E. Then $\mathscr{S}_1$ is isomorphic with the subspace $\tilde{\mathscr{S}}_1$ of $\mathscr{S}_2$ based on $\tilde{\mathscr{F}}_1=\{\tilde{\varphi}/\varphi\in\mathscr{F}_1\}$, where

$\tilde{\varphi} = \lambda sxt.\varphi(s,t)$. The spaces $\mathscr{S}_1$ and $\mathscr{S}_2$ are isomorphic whenever N is a singleton.

The proof is immediate.

Proposition 21.6. Let $\mathscr{S}_2$ be the IOS of example 21.2 and let $\mathscr{S}_1$ be obtained from example 21.1 by taking $M \times N$ for M. (Observe that whenever f_1, f_2 is a splitting scheme for M, then $\lambda sx.(f_1(s),x)$, $\lambda sx.(f_2(s),x)$ is a splitting scheme for $M \times N$.) Then $\mathscr{S}_2$ is isomorphic with the subspace $\tilde{\mathscr{S}}_2$ of $\mathscr{S}_1$ based on $\tilde{\mathscr{F}}_2 = \{\tilde{\varphi}/\varphi \in \mathscr{F}_2\}$, where $\tilde{\varphi}(s,x,t,x) = \varphi(s,x,t)$ and $\tilde{\varphi}(s,x,t,y) = \perp$ otherwise.

This isomorphism is quite immediate, too. For instance,

$$\begin{aligned}\tilde{\varphi}\tilde{\psi}(s,x,t,x) &= \sup_{r,z} \inf\{\tilde{\varphi}(s,x,r,z), \tilde{\psi}(r,z,t,x)\} \\ &= \sup_{r} \inf\{\tilde{\varphi}(s,x,r,x), \tilde{\psi}(r,x,t,x)\} \\ &= \sup_{r} \inf\{\varphi(s,x,r), \psi(r,x,t)\} = \varphi\psi(s,x,t)\end{aligned}$$

and $\tilde{\varphi}\tilde{\psi}(s,x,t,y) = \perp$ whenever $y \neq x$, hence $\tilde{\varphi}\tilde{\psi} = (\varphi\psi)\tilde{}$.

Following Skordev [1980], binary fuzzy relations can be intuitively interpreted as semantical counterparts of programs processed by a computer (or a man). Members of M are data and $\varphi(s,t) = d$ means that, given an input s, a part t is produced among the outputs; the extent of accuracy is evaluated by d. If $\varphi(s,t) = \top$, then t is fully produced, while $\varphi(s,t) = \perp$ would mean that nothing of t appears in the outputs at all. The initial IOS-operations correspond to certain natural program constructs, while $\varphi \leq \psi$ means that 'ψ is better than φ'.

The above interpretation is not the only possible one; in fact, there are as many as four pairwise dual interpretations. (Compare with the dual spaces in exercise 19.4.) First, inputs and outputs may be interchanged, with $\varphi(s,t) = d$ meaning that t is processed by φ and a part of s appears among the results. Secondly, the partial ordering of E could be viewed in another way. Namely, whenever $\varphi(s,t) = d$, $\varphi_1(s,t) = d_1$ and $d \leq d_1$, then the former computation may be regarded as providing more information than the latter, so that $\varphi(s,t) = \perp$ would mean that the result, respectively t or s, is produced entirely. Thus the motto of the corresponding two interpretations is 'The less information the better', which seems to make some sense nowadays.

Example 21.2 involves parameters, i.e. additional sources of information which is not liable to change. Examples 21.1, 21.2 can be slightly generalized by introducing a kind of complexity measure for data processing. For instance, if s is processed into t for time p_1 and the time needed to process t into r is p_2, then it is reasonable to assume that s is processed into r for time $p_1 + p_2$. If, however, p_1 and p_2 are bites of storage needed for the corresponding computations, then it would be reasonable to assume that $p_1 + p_2$ stands for the maximum rather than the sum of p_1, p_2. Of course, one may measure both these quantities by means of pairs of time and space. This suggests that complexity of data processing could be measured by members of certain semigroups as done in Skordev [1980]. A 'complexity' version of example 21.1 follows; example 21.2 may be modified similarly.

Proposition 21.7 (Example 21.3). Let S be a semigroup with a zero 0. Take $\mathscr{F} = \{\varphi/\varphi: M \times S \times M \to E\}$, $\varphi \leq \psi$ iff $\varphi(s,p,t) \leq \psi(s,p,t)$ for all $s,t \in M$ and $p \in S$, $\varphi\psi(s,p,t) = \sup_{r,p_1+p_2=p} \inf\{\varphi(s,p_1,r), \quad \psi(r,p_2,t)\}$, $(\varphi,\psi)(f_1(s),p,t) = \varphi(s,p,t)$, $(\varphi,\psi)(f_2(s),p,t) = \psi(s,p,t)$ and $(\varphi,\psi)(s,p,t) = \bot$ otherwise, $I(s,0,s) = \top$ and $I(s,p,t) = \bot$ otherwise, $L = \lambda spt.I(f_1(s),p,t)$ and $R = \lambda spt.I(f_2(s),p,t)$. Then $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is a $(**)_0$, $(***)_0$-complete OS.

Proof. The verifications of 21.1 may be repeated once again, *mutatis mutandis*. We have:

$$\begin{aligned}
\varphi(\psi\chi)(s,p,t) &= \sup_{r,p_1+p_2=p} \inf\{\varphi(s,p_1,r), \psi\chi(r,p_2,t)\} \\
&= \sup_{r,p_1+p_2=p} \inf\{\varphi(s,p_1,r), \sup_{u,p_3+p_4=p_2} \inf\{\psi(r,p_3,u), \chi(u,p_4,t)\}\} \\
&= \sup_{r,p_1+p_2=p}\ \sup_{u,p_3+p_4=p_2} \inf\{\varphi(s,p_1,r), \inf\{\psi(r,p_3,u), \chi(u,p_4,t)\}\} \\
&= \sup_{r,u,p_1+p_3+p_4=p} \inf\{\varphi(s,p_1,r), \psi(r,p_3,u), \chi(u,p_4,t)\} \\
&= \sup_{u,p_2+p_4=p}\ \sup_{r,p_1+p_3=p_2} \inf\{\inf\{\varphi(s,p_1,r), \psi(r,p_3,u)\}, \chi(u,p_4,t)\} \\
&= \sup_{u,p_2+p_4=p} \inf\{\varphi\psi(s,p_2,u), \chi(u,p_4,t)\} = (\varphi\psi)\chi(s,p,t)
\end{aligned}$$

for all s,p,t; hence $\varphi(\psi\chi) = (\varphi\psi)\chi$. Taking $L_1 = \lambda spt.I(f_1(t),p,s)$ and $R_1 = \lambda spt.I(f_2(t),p,s)$, we get $LL_1 = RR_1 = I$, $LR_1 = RL_1 = O$ and $(\varphi,\psi) = \sup\{L_1\varphi, R_1\psi\}$. Any subset $\mathscr{H}$ of $\mathscr{F}$ has a least upper bound $\sup\mathscr{H} = \lambda spt.\sup\{\theta(s,p,t)/\theta \in \mathscr{H}\}$ and multiplication is continuous with respect to least upper bounds, which completes the proof by 19.4.

Proposition 21.8. Let $\mathscr{S}$ be the IOS of example 21.3 or a subspace of it. Then

$$\begin{aligned}
\langle\varphi\rangle(f_2^n(f_1(s)), p, f_2^n(f_1(t))) &= \varphi(s,p,t), \\
\Delta(\varphi,\psi)(f_2^n(f_1(s)), p, t) &= \varphi\psi^n(s,p,t)
\end{aligned}$$

and $\langle\varphi\rangle(s,p,t)$, $\Delta(\varphi,\psi)(s,p,t) = \bot$ otherwise,

$$\begin{aligned}
[\varphi](s,p,t) = \sup\{\inf_{i<n} \varphi(f_2^{-1}(r_i), p_i, r_{i+1})/n \in \omega \,\&\, r_0 = s \,\&\, p_0 + \cdots + p_{n-1} = p \\
\&\, r_0, \ldots, r_{n-1} \in f_2(M) \,\&\, r_n = f_1(t)\}.
\end{aligned}$$

The proof follows that of 21.2.

Proposition 21.9. Let $\mathscr{S}_1, \mathscr{S}_3$ be the IOS respectively of examples 21.1, 21.3. Then $\mathscr{S}_1$ is isomorphic to the subspace $\tilde{\mathscr{S}}_1$ of $\mathscr{S}_3$ based on $\tilde{\mathscr{F}}_1 = \{\tilde{\varphi}/\varphi \in \mathscr{F}_1\}$, where $\tilde{\varphi}(s,0,t) = \varphi(s,t)$ and $\tilde{\varphi}(s,p,t) = \bot$ otherwise. The spaces $\mathscr{S}_1$ and $\mathscr{S}_3$ are isomorphic whenever $S = \{0\}$.

The proof is immediate.

An interesting semigroup along with those suggested above is $S = \cup_n M^n$, where $0 = \Lambda$ and the semigroup operation is concatenation. Members of S are regarded as computation traces, partial ones in general, for I, L, R leaves no traces.

One can obtain more particular IOS from examples 21.1–21.3 by taking specific sets M augmented with splitting schemes; several options were suggested in the comments to 16.3. On the other hand, specific lattices E can be taken. For instance, any finite distributive lattice would do. The sets $\{\eta/\eta \le \xi\}$ and the intervals $[0,1]$, $[0,\infty]$ with their ordinary total orderings can also play the role of E. Some standard first order examples will be obtained this way in the next three chapters with $E = \{\perp, T\}$. (We recall that first order examples (i.e. models) are those composed of function-like elements, while higher order examples are those composed of operator-like elements.)

As far as first order examples are concerned, while the pairing scheme Π, L, R involves a splitting scheme usually realized by a *counter*, the operation $\langle\ \ \rangle$ can be implemented by means of an additional counter. To show that one needs the following Translation Elimination Theorem, a theoretical enclave in this applied part of the book.

Proposition 21.10. Let $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ be an IOS and $W, W_1, W_2 \in \mathscr{F}$ such that $W_1 W = L$, $W_2 W = R$, $W(L, R) = W$, $LW = W(L^2, LR)$ and $RW = W(RL, R^2)$. Let

$$\mathscr{C} = \{\sigma \in \mathscr{F} / W_i \sigma = \sigma W_i, \quad i = 1, 2\},$$

assuming that $L, R, (L, R), \langle I \rangle \in \mathscr{C}$. Then φ is recursive in $\{W, W_1, W_2\} \cup \mathscr{B}$ iff φ is prime recursive in $\{W, W_1, W_2\} \cup \mathscr{B}$, provided $\mathscr{B} \subseteq \mathscr{C}$.

Proof. We have $W_i(L, R) = (W_i L, W_i R)$ and the more general equalities

$$W_i(\varphi, \psi) = (W_i \varphi, W_i \psi),$$
$$LW(\varphi, \psi) = W(L\varphi, L\psi), RW(\varphi, \psi) = W(R\varphi, R\psi)$$

hold for $i = 1, 2$ and all $\varphi, \psi \in \mathscr{F}$.

The set $\mathscr{C}$ is obviously closed under $\circ, \Pi$. In order to show that it is closed under $\langle\ \ \rangle$, suppose $\sigma \in \mathscr{C}$. Then the equalities

$$\sigma L W_i = W_i \sigma L = L W_i \langle \sigma \rangle,$$
$$R W_i = W_i R, \quad W_i \langle \sigma \rangle R = R W_i \langle \sigma \rangle, \quad i = 1, 2$$

imply $\langle \sigma \rangle W_i = \langle I \rangle W_i \langle \sigma \rangle$ by 6.20. Therefore,

$$W_i \langle \sigma \rangle = W_i \langle I \rangle \langle \sigma \rangle = \langle I \rangle W_i \langle \sigma \rangle = \langle \sigma \rangle W_i, i = 1, 2;$$

hence $\langle \sigma \rangle \in \mathscr{C}$.

Our next aim is to prove that $\langle \sigma \rangle$ is prime recursive in W, W_1, W_2, σ for all $\sigma \in \mathscr{C}$, while $\langle W \rangle$, $\langle W_1 \rangle$, $\langle W_2 \rangle$ are prime recursive in W, W_1, W_2.

Using 6.13, we get

$$[W_2] W_2 = R[(W_2 L, W_2 R)] = R[W_2(L, R)] = R[W_2].$$

Writing W_0 for $W(L, R^2)$, it follows that

$$R^2[W_0] = RW_0[W_0] = W_0(RL, R^2)[W_0] = W_0(R, R^2[W_0]),$$

hence $R[W_0]R \leq R^2[W_0]$ by 6.11. Conversely,

$$\begin{aligned} W_0(I, W(I, R[W_0]R)) &= W(I, RW(I, R[W_0]R)) = W(I, W(R, R^2[W_0]R)) \\ &= W(I, W_0[W_0]R) \end{aligned}$$

gives $R[W_0] \leq W(I, R[W_0]R)$ by 6.11, hence

$$R^2[W_0] \leq RW(I, R[W_0]R) = W(R, R^2[W_0]R) = W_0[W_0]R = R[W_0]R.$$

Therefore, $R^2[W_0] = R[W_0]R$.

Using the above equalities, we get for $\sigma \in \mathscr{C}$

$$\begin{aligned} LW_1[W_2]\sigma LW_0[W_0] &= W_1\sigma LW_0[W_0] = \sigma LW_1W_0[W_0] \\ &= \sigma L^2[W_0] = \sigma L \end{aligned}$$

and

$$\begin{aligned} &RW_1[W_2]\sigma LW_0[W_0] \\ &\quad = W_1R[W_2]\sigma LW_0[W_0] = W_1[W_2]W_2\sigma LW_0[W_0] \\ &\quad = W_1[W_2]\sigma LW_2W_0[W_0] = W_1[W_2]\sigma LR^2[W_0] \\ &\quad = W_1[W_2]\sigma LR[W_0]R = W_1[W_2]\sigma LW_0[W_0]R, \end{aligned}$$

which implies $\langle\sigma\rangle = \langle I\rangle W_1[W_2]\sigma LR[W_0]$ by 6.20. This equality is to be simplified by deleting the multiplier $\langle I\rangle$.

On the one hand, $R[W_2] = [W_2]W_2$ and

$$(L[W_2], [W_2]W_2) = (I, R[W_2]) = [W_2]$$

imply $\langle I\rangle[W_2] \leq [W_2]$ by (£). On the other hand,

$$(I, W_2\langle I\rangle[W_2]) = (I, \langle I\rangle W_2[W_2]) = (L, \langle I\rangle R)[W_2] = \langle I\rangle[W_2]$$

implies $[W_2] \leq \langle I\rangle[W_2]$ by (££). Therefore,

$$\begin{aligned} \langle\sigma\rangle &= \langle I\rangle W_1[W_2]\sigma LR[W_0] = W_1\langle I\rangle[W_2]\sigma LR[W_0] \\ &= W_1[W_2]\sigma LR[W_0] \end{aligned}$$

for all $\sigma \in \mathscr{C}$. In particular, $\langle I\rangle$ is prime recursive in W, W_1, W_2.

The equalities $W_1L = LW_1$, $W_1R = RW_1$ give $\langle W_1\rangle = \langle I\rangle W_1$ by 6.20 and similarly $\langle W_2\rangle = \langle I\rangle W_2$.

The equalities

$$\begin{aligned} &LW(\langle L\rangle, \langle R\rangle) = W(L\langle L\rangle, L\langle R\rangle) = W(L^2, RL) = W(L, R)L = WL, \\ &RW(\langle L\rangle, \langle R\rangle) = W(R\langle L\rangle, R\langle R\rangle) = W(\langle L\rangle, \langle R\rangle)R \end{aligned}$$

imply $\langle W\rangle = \langle I\rangle W(\langle L\rangle, \langle R\rangle)$ by 6.20.

Suppose that $\mathscr{B} \subseteq \mathscr{C}$ and φ is recursive in $\{W, W_1, W_2\} \cup \mathscr{B}$. Then φ is prime recursive in $\mathscr{B}_1 = \{\langle B\rangle, \langle\langle L\rangle\rangle, \langle\langle A\rangle\rangle, \langle W\rangle, \langle W_1\rangle, \langle W_2\rangle\} \cup \langle\mathscr{B}\rangle$ by 7.11. Noting that B, $\langle L\rangle$, $\langle A\rangle \in \mathscr{C}$, we conclude that all the members of

$\mathscr{B}_1$ are prime recursive in $\{W, W_1, W_2\} \cup \mathscr{B}$, hence so is φ. This completes the proof.

The elements W, W_1, W_2 seem to express axiomatically the presence of an extra counter besides that implied by Π, L, R. Example 21.1 illustrates this, while other first order examples can be treated similarly.

Proposition 21.11. Let $\mathscr{S}_0 = (\mathscr{F}_0, I_0, \Pi_0, L_0, R_0)$ be the IOS of example 21.1 and let $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ be obtained from the same example by taking $M \times \omega$ for M, i.e. by adding a new counter. To each $\varphi_0 \in \mathscr{F}_0$ assign an element $\tilde{\varphi}_0 \in \mathscr{F}$ such that $\tilde{\varphi}_0(s,n,t,n) = \varphi_0(s,t)$ and $\tilde{\varphi}_0(s,m,t,n) = \perp$ otherwise. Take $W_1(s,n,s,2n)$, $W_2(s,n,s,2n+1) = \top$ and $W_1(s,m,t,n)$, $W_2(s,m,t,n) = \perp$ otherwise, $W(s,2m,t,n) = L(s,m,t,n)$ and $W(s,2m+1,t,n) = R(s,m,t,n)$. Let $\varphi \in \mathscr{F}$, $\mathscr{B}_0 \subseteq \mathscr{F}_0$. Then φ is recursive in $\{W, W_1, W_2\} \cup \mathscr{B}_0^{\sim}$ iff φ is prime recursive in $\{W, W_1, W_2\} \cup \mathscr{B}_0^{\sim}$. If $\varphi_0 \in \mathscr{F}_0$ is recursive in $\mathscr{B}_0$, then $\varphi_0^{\sim}$ is prime recursive in $\{W, W_1, W_2\} \cup \mathscr{B}_0^{\sim}$.

Proof. The elements W, W_1, W_2 satisfy the corresponding equalities of 21.10 and all $\tilde{\psi}_0$ commute with W_1, W_2. Therefore, 21.10 implies that φ is recursive in $\{W, W_1, W_2\} \cup \mathscr{B}_0^{\sim}$ iff it is prime recursive in the same set.

It follows from 21.5, 21.6 that $\tilde{\mathscr{S}}_0 = (\tilde{\mathscr{F}}_0, I, \Pi \restriction \tilde{\mathscr{F}}_0^2, L, R)$ is a subspace of $\mathscr{S}$ isomorphic to $\mathscr{S}_0$. Whenever $\varphi_0 \in \mathscr{F}_0$ is recursive in $\mathscr{B}_0$, then $\tilde{\varphi}_0$ is recursive in $\mathscr{B}_0^{\sim}$, hence $\varphi_0^{\sim}$ is prime recursive in $\{W, W_1, W_2\} \cup \mathscr{B}_0^{\sim}$. The proof is complete.

The IOS of examples 21.1–21.3 have some nice properties not shared by spaces considered in chapter 25 below. Least fixed points of inductive mappings are reached at level ω according to the proofs of 18.2, 18.14. Collection operations can be introduced by exercise 18.5, while 19.6 implies that there are elements U satisfying condition (1) of exercise 7.10. In the case of example 21.1 $Co\{\varphi_n\}(f_2^n(f_1(s)), t) = \varphi_n(s,t)$ and $Co\{\varphi_n\}(s,t) = \perp$ otherwise, $U(s, f_1(s))$, $U(s, f_2(s)) = \top$ and $U(s,t) = \perp$ otherwise. Observe that $U(\varphi, \psi) = \sup\{\varphi, \psi\}$ for all φ, ψ.

The following two statements introduce certain t-operations in example 21.1, while examples 21.2, 21.3 are treated similarly.

Proposition 21.12. Let $\mathscr{S}$ be the IOS of example 21.1 and $\mathrm{In}(\varphi) = \varphi^{-1} = \lambda st.\varphi(t,s)$ (*inversion*). Then $\langle\ \rangle = \lambda\varphi.(\varphi, In(\varphi))$ is a t-operation with a corresponding set of functional elements $\mathscr{B}_0 = \{U\}$ such that $\mathscr{S}, \langle\ \rangle$ satisfy the axiom $\mathrm{t}\mu\mathrm{A}_3$.

Proof. While our intention is to add In to the initial IOS-operations, we take $\langle\ \rangle$ instead in order to ensure that φ is recovered from $\langle\varphi\rangle$ recursively. We have $\varphi = L\langle\varphi\rangle$ and $\varphi^{-1} = R\langle\varphi\rangle$. Using the easy equalities $L_1^{-1} = L$, $R_1^{-1} = R$, $(\varphi\psi)^{-1} = \psi^{-1}\varphi^{-1}$, $(\varphi^{-1})^{-1} = \varphi$ and $(\sup\{\varphi,\psi\})^{-1} = \sup\{\varphi^{-1}, \psi^{-1}\}$, we get

$$\begin{aligned}\langle\varphi\psi\rangle &= (\varphi\psi, (\varphi\psi)^{-1}) = (\varphi\psi, \psi^{-1}\varphi^{-1}) \\ &= (L\langle\varphi\rangle L\langle\psi\rangle, R\langle\psi\rangle R\langle\varphi\rangle),\end{aligned}$$

$$\begin{aligned}\langle\!\langle(\varphi,\psi)\rangle\!\rangle &= ((\varphi,\psi),(\sup\{L_1\varphi,R_1\psi\})^{-1}) = ((\varphi,\psi),\sup\{\varphi^{-1}L,\psi^{-1}R\})\\ &= ((L\langle\!\langle\varphi\rangle\!\rangle, L\langle\!\langle\psi\rangle\!\rangle), U(R\langle\!\langle\varphi\rangle\!\rangle L, R\langle\!\langle\psi\rangle\!\rangle R)),\\ \langle\!\langle\langle\!\langle\varphi\rangle\!\rangle\rangle\!\rangle &= (\langle\!\langle\varphi\rangle\!\rangle,(\sup\{L_1\varphi,R_1\varphi^{-1}\})^{-1})\\ &= (\langle\!\langle\varphi\rangle\!\rangle, U(\varphi^{-1}L,\varphi R)) = (\langle\!\langle\varphi\rangle\!\rangle, U(R\langle\!\langle\varphi\rangle\!\rangle L, L\langle\!\langle\varphi\rangle\!\rangle R)).\end{aligned}$$

The operation In is continuous with respect to least upper bounds of arbitrary sets, hence so is $\langle\!\langle\ \rangle\!\rangle$. Therefore, $\mathscr{S}$, $\langle\!\langle\ \rangle\!\rangle$ satisfy both conditions $(\mathrm{t}{*}{*})_0$ and $(\mathrm{t}{*}{*}{*})_0$, hence $\mathrm{t}\mu\mathrm{A}_3$ is valid by 18.20 or 18.22. Using 10.9*, 10.10*, we get that $\langle\!\langle\ \rangle\!\rangle$ is a t-operation, which completes the proof.

Proposition 21.13. Let $\mathscr{S}$ be the IOS of example 21.1, $\varnothing \subset M_0 \subseteq M$, let $J: M_0 \times M \to M$ is injective and $J(M_0^2) \subseteq M_0$. Take

$$St(\varphi)(J(s,t),J(s,r)) = \varphi(t,r)$$

and $St(\varphi)(s,t) = \bot$ otherwise, and $\langle\!\langle\ \rangle\!\rangle = \lambda\varphi.(\varphi, St(\varphi))$. Then St is a storing operation and $\mathscr{S}$, $\langle\!\langle\ \rangle\!\rangle$ satisfy the axiom $\mathrm{t}\mu\mathrm{A}_3$.

Proof. Take $\mathscr{L} = M_0^{\sim} = \{\tilde{s}/s\in M_0\}$, where $\tilde{s} = \lambda tr.I(J(s,t),r)$, $K_0(J(s,f_i(t)), f_i(J(s,t))) = \top$, $i=1,2$, and $K_0(s,t) = \bot$ otherwise. Then

$$\tilde{s}K_0(f_1(t),r) = I(f_1(J(s,t)),r) = \tilde{s}L(t,r) = (\tilde{s}L,\tilde{s}R)(f_1(t),r)$$

and $\tilde{s}K_0(f_2(t),r) = \tilde{s}R(t,r) = (\tilde{s}L,\tilde{s}R)(f_2(t),r)$, while $\tilde{s}K_0(t,r)$, $(\tilde{s}L,\tilde{s}R)(t,r) = \bot$ otherwise. Therefore, $\tilde{s}K_0 = (\tilde{s}L,\tilde{s}R)$.

Taking $K_1(J(s,J(t,r)), J(J(s,t),r)) = \top$, $K_1(s,t) = \bot$ otherwise, and $K_2 = K_1^{-1}$, one easily obtains $\tilde{s}\tilde{t}K_1 = J(t,s)^{\sim}$ and $J(t,s)^{\sim}K_2 = \tilde{s}\tilde{t}$.

The equality $\tilde{s}St(\varphi) = \varphi\tilde{s}$ follows from the definition of St, while $\forall\tilde{s}(\tilde{s}\varphi \leq \tilde{s}\psi)$ says exactly that $St(I)\varphi \leq St(I)\psi$; hence the axiom ($) holds by exercise 10.7. Therefore, St is a storing operation, which implies by 10.17 that $\langle\!\langle\ \rangle\!\rangle$ is a t-operation. The latter is obviously continuous with respect to least upper bounds of arbitrary sets, hence $\mathscr{S}$, $\langle\!\langle\ \rangle\!\rangle$ satisfy conditions $(\mathrm{t}{*}{*})_0$, $(\mathrm{t}{*}{*}{*})_0$, each of which ensures the validity of the axiom $\mathrm{t}\mu\mathrm{A}_3$. This completes the proof.

Remark. To avoid trivialities M_0 is assumed not a singleton, hence $\mathrm{Card}(M_0) \geq \omega$ by $J(M_0^2) \subseteq M_0$. If $M_0 = M$, then we have a pairing function for M as in exercise 10.1. If $M_0 \subset M$, then we have a *restricted pairing function* for M as in Moschovakis [1971].

Restricted pairing functions appear naturally when Cartesian products are considered. For instance, let $J_0: N_0^2 \to N_0$ be injective, let N be a nonempty set and let $M = N_0 \times N$. Take a fixed $r_0 \in N$, $M_0 = N_0 \times \{r_0\}$ and $J((s,r_0),(t,r)) = (J_0(s,t),r)$ for all $s,t\in N_0$ and $r\in N$. Then M, M_0 and J will satisfy the assumptions of 21.13. (If a restricted pairing function for N_0 is given, then one nevertheless gets a restricted pairing function for $M = N_0 \times N$ this way.) A possible splitting scheme for this particular M is $f_1 = \lambda sr.(J_0(s_1,s),r)$, $f_2 = \lambda sr.(J_0(s_2,s),r)$ with certain fixed $s_1 \neq s_2 \in N_0$. If another splitting scheme for M has already been chosen, then simple transition between the corresponding pairing schemes for $\mathscr{F}$ are provided by exercises 4.6, 7.2.

It should be stressed that St is itself a t-operation by 10.18 since $I = K_3K_4$ for any $K_3 \in M_0^{\sim}$ and $K_4 = K_3^{-1}$ or $K_4(J(s,t),t) = \top$ and $K_4(s,t) = \bot$ otherwise.

The notions of t-recursiveness which correspond to the operations $\langle\!\langle\ \rangle\!\rangle$ of 21.12, 21.13 are called respectively in-*recursiveness*, st-*recursiveness*. The following assertion shows that these t-operations can be combined into a single t-operation, which allows both In, St to be added to the initial IOS-operations. The notion of t-recursiveness thus introduced is called st, in-*recursiveness*.

Proposition 21.14. Let $\mathscr{S}$ be the IOS of example 21.1 and let $\langle\!\langle\ \rangle\!\rangle$, $\langle\!\langle\ \rangle\!\rangle^*$ be introduced respectively by 21.12, 21.13. Then $\lambda\varphi.\langle\!\langle\langle\!\langle\varphi\rangle\!\rangle^*\rangle\!\rangle$ is a t-operation satisfying the axiom $\mathrm{t}\mu\mathrm{A}_3$ and so is $\lambda\varphi.\langle\!\langle\langle\!\langle\varphi\rangle\!\rangle\rangle\!\rangle^*$.

Proof. We have

$$St(\varphi^{-1})(J(s,t), J(s,r)) = \varphi^{-1}(t,r) = \varphi(r,t) = St(\varphi)^{-1}(J(s,t), J(s,r))$$

and $St(\varphi^{-1})(s,t)$, $St(\varphi)^{-1}(s,t) = \perp$ otherwise, hence $St(\varphi^{-1}) = St(\varphi)^{-1}$. Using this equality, 10.12 and 10.13, $\langle\!\langle\langle\!\langle\varphi\rangle\!\rangle\rangle\!\rangle^*$ is expressed by $\langle\!\langle\langle\!\langle\varphi\rangle\!\rangle^*\rangle\!\rangle$ and vice versa. Therefore, both $\lambda\varphi.\langle\!\langle\langle\!\langle\varphi\rangle\!\rangle^*\rangle\!\rangle$ and $\lambda\varphi.\langle\!\langle\langle\!\langle\varphi\rangle\!\rangle\rangle\!\rangle^*$ are t-operations by exercise 10.6. They are continuous with respect to least upper bounds, which implies $\mathrm{t}\mu\mathrm{A}_3$ by 18.22. The proof is complete.

The last three statements show that, leaving c-recursiveness to one side, there are at least four different concepts of effective computability for E-valued fuzzy relations: recursiveness, in-recursiveness, st-recursiveness and st, in-recursiveness. Of course, they are effective only if the splitting and pairing schemes involved are effective. A number of properties of these notions follow directly from the general theory developed in chapters 7–10.

It will be shown in chapter 24 that the notion of st-recursiveness with $M_0 = M$ generalizes the prime computability of Moschovakis. Adding U to the initial elements, one gets a natural broader concept of effective computability which in a sense generalizes Friedman's computability by effectively definitional schemes. The replacement of U by $U^* = \lambda st.\top$ results in a broadest notion which generalizes the search computability of Moschovakis and which, as will be shown in the exercises below, extends in-recursiveness as well.

EXERCISES TO CHAPTER 21

Exercise 21.1. Let $\mathscr{S}$ be the IOS of example 21.1 or 21.2, 21.3. Show that $(L,R) \leq I$, while $(L,R) = I$ iff $f_1(M) \cup f_2(M) = M$, and $\langle I\rangle = I$ iff $\bigcup_n f_2^n(f_1(M)) = M$.

Hint. Use exercise 16.5.

Comments. While $\bigcup_n f_2^n(f_1(M)) = M$ implies $f_1(M) \cup f_2(M) = M$, the reverse implication fails in example 22.9.

Exercise 21.2 (Example 21.4). Let $\mathscr{S}_0 = (\mathscr{F}_0, I, \Pi_0, L, R)$ be the IOS of example 21.1 and let $\mathscr{F} = \{\varphi / \forall str(\varphi(s,t), \varphi(s,r) \neq \perp \Rightarrow r = t)\}$. Show that $\mathscr{S} = (\mathscr{F}, I, \Pi_0 \restriction \mathscr{F}^2, L, R)$ is a $(**)_0$, $(***)_0$-complete subspace of $\mathscr{S}_0$ and if $\langle\!\langle\ \rangle\!\rangle$ is an operation over $\mathscr{F}_0$ introduced by 21.13, then $\langle\!\langle\ \rangle\!\rangle \restriction \mathscr{F}$ is a t-operation over $\mathscr{F}$ which meets $\mathrm{t}\mu\mathrm{A}_3$.

Hint. Use 18.15, 18.16 and a t-analog to 18.16. The elements K_0, K_1, K_2 of the proof of 21.13 are in $\mathscr{F}$.

Similar assertions hold for example 21.2, 21.3; in the latter case one may also take

$$\mathscr{F} = \{\varphi/\forall spp_1 \mathrm{tr}(\varphi(s,p,t), \varphi(s,p_1,r) \neq \perp \Rightarrow p_1 = p \& r = t)\}.$$

Exercise 21.3. Let $\mathscr{S}$ be the IOS of example 21.3. Specify a semi group S and elements φ, ψ such that φ is polynomial in ψ but neither strictly polynomial in ψ nor strictly primitive in ψ.

Hint. Take $S = (\omega, +)$, $\psi(s, 1, s) = \top$ and $\psi(s, p, t) = \perp$ otherwise, then take $\varphi = \psi^2$.

Exercise 21.4. Let $W, W_1, W_2, \mathscr{C}$ be the same as in 21.10, $\mathscr{C}_1 = \{\sigma/L\sigma = \sigma L \& R\sigma = \sigma R\}$ and $\mathscr{C}_2 = \{\sigma/L\sigma = \sigma B^2 \& R\sigma = \sigma A^2 \& \sigma(L, R) = \sigma\}$. Show that if $\mathscr{B} \subseteq \mathscr{C} \cup \mathscr{C}_1 \cup \mathscr{C}_2$ and $\mathscr{B}_1 \subseteq \mathscr{F}$, then φ is recursive in $\{W, W_1, W_2\} \cup \mathscr{B} \cup \mathscr{B}_1$ iff it is prime recursive in $\{W, W_1, W_2\} \cup \mathscr{B} \cup \langle \mathscr{B}_1 \rangle$.

Hint. Following the proof of 21.10, show that $\langle \sigma \rangle = \langle I \rangle \sigma$ for all $\sigma \in \mathscr{C}_1$, while $\langle \sigma \rangle = \langle I \rangle \sigma(\langle L \rangle, \langle R \rangle)$ for all $\sigma \in \mathscr{C}_2$.

Notice that $\mathscr{C}_1 L \subseteq \mathscr{C}_2$ and the elements U, V of exercise 7.10, 7.14, if any, are in $\mathscr{C}_2$.

Exercise 21.5. Let $\mathscr{S}_0, \mathscr{S}$ be the IOS of 21.11, let $\mathscr{B} \subseteq \mathscr{F}$, $\mathscr{B}^* \subseteq \mathscr{F}_0$ and let a $\psi^* \in \mathscr{B}^*$ correspond to each $\psi \in \mathscr{B}$ such that $\psi^*(\bar{m}(s), \bar{n}(t)) = \psi(s, m, t, n)$, and whenever t is not of the form $\bar{n}(r)$, then $\psi^*(\bar{m}(s), t) = \perp$, where $\bar{m}(s)$ stands for $f_2^m(f_1(s))$. Let $Z(s, m, s, (2m+1)\mathrm{sg}m) = \top$ and $Z(s, m, t, n) = \perp$ otherwise. Show that for every $\varphi \in \mathscr{F}$ recursive in $\{W, W_1, W_2, Z\} \cup \mathscr{B}$ there is a $\varphi^* \in \mathscr{F}_0$ recursive in $\mathscr{B}^*$ corresponding to φ as above.

Hint. Take $L^* = \langle L_0 \rangle$, $R^* = \langle R_0 \rangle$. There is a recursive element $\varphi_0 \in \mathscr{F}_0$ such that $\bar{n}\varphi_0 = \overline{2n}$; take $W_1^* = \varphi_0$. Similarly W_2^*, W^*, Z^* are constructed. If φ^*, ψ^* correspond to φ, ψ, take $(\varphi\psi)^* = \varphi^*\psi^*$, $(\varphi, \psi)^* = C_0(\varphi^*, \psi^*)$, $\langle \varphi \rangle^* = G_0 \langle \varphi^* \rangle G_0$ and $[\varphi]^* = C_0[\varphi^* C_0]$, where C_0, G_0 are the elements of $\mathscr{F}_0$ introduced in 6.35 and exercise 6.4.

Exercise 21.6. Let $\mathscr{S}_0, \mathscr{S}$ be the IOS of 21.11, $\varphi_0 \in \mathscr{F}_0$ and $\mathscr{B}_0 \subseteq \mathscr{F}_0$. Show that if $\tilde{\varphi}_0$ is recursive in $\{W, W_1, W_2, Z\} \cup \tilde{\mathscr{B}_0}$, then φ_0 is recursive in $\mathscr{B}_0$.

Hint. Take $\tilde{\mathscr{B}}_0^* = \langle \mathscr{B}_0 \rangle$ and get by exercise 21.5 an element $\tilde{\varphi}_0^*$ recursive in $\langle \mathscr{B}_0 \rangle$ to correspond to $\tilde{\varphi}_0$. Then φ_0 will be recursive in $\mathscr{B}_0$ since $\varphi_0 = L_0 \tilde{\varphi}_0^*(I_0, I_0)$.

Remarks. Combined with 21.11, the last two exercises show that translation is equivalent to counting. More generally, the storing operation in first order contexts is connected with computations on complex storages and can be implemented by stacking. The latter can be established by using the fact that $St(\sigma) = St(I)S_1\sigma S_2$, provided $xS_1S_2 = x$ and $xS_1\sigma = \sigma x S_1$ for all $x \in \mathscr{L}$. However, we shall not burden the exposition with further (though interesting) details.

Exercise 21.7. Show that each of the operations $\langle\!\langle\ \ \rangle\!\rangle$, St of 21.13 satisfies the assumptions of exercises 10.2–10.5. Take $K_5 = \lambda st.I(J(s,s),t)$, $K_4(J(s,t),t) = \top$, $K_6(J(s,J(t,r)), J(t,J(s,r))) = \top$ and $K_4(s,t)$, $K_6(s,t) = \bot$ otherwise, and show that the assumptions of exercise 10.9 are also satisfied.

Hint. Use exercises 10.8**, 18.7.

Remarks. According to exercise 21.7, $\langle\ \ \rangle$ is always expressible in terms of the operation St of 21.13. On the other hand, whenever M_0 is countable, then St can be expressed in terms of $\langle\ \ \rangle$ and st-recursiveness reduces to recursiveness. It follows easily from 10.20 that operations St based on wider sets M_0 are more powerful. In particular, all such operations can be expressed in terms of one which corresponds to $M_0 = M$.

Exercise 21.8. Let $M_0 = M$. Show that the operation $\lambda\varphi\psi.\inf\{\varphi,\psi\}$ can be expressed by St and certain elements.

Hint. Show that $\inf\{\varphi,\psi\} = K_5 St(\varphi) K_1^* St(\psi) K_5^{-1}$, where $K_1^*(J(s,t), J(t,s)) = \top$ and $K_1^*(s,t) = \bot$ otherwise.

Exercise 21.9. Let $M_0 = M$. Show that the operation In can be expressed in terms of St and certain elements.

Hint. Show that

$$\varphi^{-1} = K_5 St(U^* K_5) K_6 St(St(\varphi) K_5^{-1}) K_1^* K_4.$$

Example 21.5. Example 21.2 with $N = M^{n-1}$, $n > 0$. Therefore, $\mathscr{F} = \{\varphi/\varphi : M^{n+1} \to E\}$ consists of $n+1$-ary E-valued fuzzy relations.

Exercise 21.10. Let $\mathscr{S}$ be the IOS of example 21.5, let $J : M^2 \to M$ be injective and, writing $J(s_1,\ldots,s_{n+1})$ for $J(s_1, J(s_2,\ldots,s_{n+1}))$, $St(\varphi)(J(s_1,t_1,\ldots,t_n), s_2,\ldots,s_n, J(s_1,r,t_2,\ldots,t_n)) = \varphi(t_1,\ldots,t_n,r)$ and $St(\varphi)(s_1,\ldots,s_n,r) = \bot$ otherwise. Prove that St is a storing operation in the sense of exercise 10.10 and $\langle\!\langle\ \ \rangle\!\rangle = \lambda\varphi.(\varphi, St(\varphi))$ is a t-operation satisfying $t\mu A_3$.

Hint. Take $\mathscr{L} = \widetilde{M^n}$, where

$$(s_1,\ldots,s_n)\tilde{}(\varphi) = \lambda t_1 \ldots t_n r.\varphi(J(s_1,t_1,\ldots,t_n), s_2,\ldots,s_n,r).$$

While the multiplication of $\mathscr{S}$ is a particular instance of composition of $n+1$-ary fuzzy relations, the operation St makes it possible to fully work out such compositions.

Exercise 21.11. Let $\mathscr{S}$ be the IOS of example 21.5. Show that the mapping

$$\lambda\varphi\psi_1\ldots\psi_n.\lambda s_1\ldots s_n t.\sup_{r_1,\ldots,r_n}\ \inf_{1\le i\le n}\{\psi_i(s_1,\ldots,s_n,r_i), \varphi(r_1,\ldots,r_n,t)\}$$

is st-recursive in certain elements.

Exercise 21.12 (Example 21.6). Let $\mathscr{S}_0 = (\mathscr{F}_0, I, \Pi_0, L, R)$ be the IOS of example 21.1, let $\lesssim$ be a partial quasi-ordering of M (i.e., a reflexive and transitive binary relation on M) and let $J : M^2 \longrightarrow M$ be injective, such that $f_1(s) \approx f_2(s) \approx s$ (i.e., both $\lesssim$ and $\gtrsim$ hold). Suppose also that, if $J(s,r)\downarrow$ and

$t \lesssim r$, then $J(s,t)\downarrow$ and $J(s,t)$, $r \lesssim J(s,r)$, and if $J(s, J(t,r))\downarrow$, then $J(J(s,t),r)\downarrow$ and $J(s, J(t,r)) \approx J(J(s,t),r)$ for all $s,t,r \in M$. Take the subspace $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ of $\mathscr{S}_0$ based on $\mathscr{F} = \{\varphi / \forall st(\varphi(s,t) \neq \perp \Rightarrow s \lesssim t)\}$, and take $St(\varphi)(J(s,t), J(s,r)) = \varphi(t,r)$ and $St(\varphi)(s,t) = \perp$ otherwise. Show that St is a storing operation and $\langle\!\langle\ \ \rangle\!\rangle = \lambda\varphi.(\varphi, St(\varphi))$ is a t-operation satisfying $\mathrm{t}\mu\mathrm{A}_3$.

Hint. Take $\mathscr{L} = \{\tilde{s} / \exists t(J(s,t)\downarrow)\}$, where $\tilde{s}(t,r) = \top$, if $r = J(s,t)$, and $\tilde{s}(t,r) = \perp$ otherwise. Follow the proof of 21.13.

Remark. Proposition 21.13 corresponds to $\lesssim \, = M^2$. It cannot now be claimed that $I \in \mathscr{L} \circ \mathscr{F}$, St is a t-operation and is sufficient to express $\langle\ \ \rangle$; here is an example illustrating this.

Exercise 21.13. Let $\mathscr{S}_0$ be the space of example 21.1 with $M_n^* = \omega \times \bigcup_{0<i\leq n} M^i$ taken for M, the splitting scheme operating on ω, $x \lesssim y$ iff $\mathrm{lh}(x) \leq lh(y)$, where $lh(k, s_1, \ldots, s_m) = m$, and let $J: M_n^{*2} \dashrightarrow M_n^*$ be introduced as suggested in the remark preceding 16.5 so that $J(x,y)\downarrow$ iff $lh(x) + lh(y) \leq n$. Show that $\lesssim$, J satisfy the assumptions of exercise 21.12.

If M_n^* is defined as $\omega \times \bigcup_{i\leq n} M^i$, then $I \in \mathscr{L} \circ \mathscr{F}$ and $\langle\ \ \rangle$ is expressible, but still no elements K_4, K_5 can be specified to meet the requirements of exercise 10.9.

CHAPTER 22

Number functions and relations

This chapter studies in detail some standard examples which consist of functions and relations on natural numbers or sequences of natural numbers. The classical notions of recursive enumerability, relative μ-recursiveness and relative partial recursiveness are accommodated, establishing another connection with Ordinary Recursion Theory in addition to representability.

The starting point is examples 21.1, 21.2 with $\{\bot, \top\}$ taken for E. While these spaces consist of relations, i.e. multiple-valued functions, the single-valued functions are shown to form respective subspaces.

Example 22.1. Example 21.1 with $E = \{\bot, \top\}$.

Proposition 22.1 (Example 22.2). Let $\mathscr{S}_0 = (\mathscr{F}_0, I, \Pi_0, L, R)$ be the IOS of example 22.1 and

$$\mathscr{F} = \{\varphi / \forall str(\varphi(s,t), \varphi(s,r) = \top \Rightarrow r = t)\}.$$

Then $\mathscr{S} = (\mathscr{F}, I, \Pi_0 \restriction \mathscr{F}^2, L, R)$ is a $(**)_0$, $(***)_0$-complete subspace of $\mathscr{S}_0$.

Proof. It is immediate that $I, L, R \in \mathscr{F}$ and it easily follows that $\mathscr{F}$ is closed under multiplication and pairing, hence $\mathscr{S}$ is a subspace of $\mathscr{S}_0$ as an OS.

Suppose that $\mathscr{H}$ is a well ordered subset of $\mathscr{F}$ and $\varphi = \sup \mathscr{H}$ in $\mathscr{F}_0$. If $\varphi(s,t), \varphi(s,r) = \top$, then there is a $\theta \in \mathscr{H}$ such that $\theta(s,t), \theta(s,r) = T$, hence $r = t$. Therefore, $\varphi \in \mathscr{F}$, which completes the proof by 18.15, 18.16.

Later, members of $\mathscr{F}_0$ will be regarded as binary relations by interpreting $\varphi(s,t) = \bot$, $\varphi(s,t) = \top$ respectively as $(s,t) \notin \varphi$, $(s,t) \in \varphi$. Therefore, $\mathscr{F} = \{\varphi / \varphi : M \dashrightarrow M\}$, $\varphi \leq \psi$ iff $\varphi \subseteq \psi$, $\varphi\psi = \lambda s.\psi(\varphi(s))$, $(\varphi, \psi)(f_1(s)) = \varphi(s)$, $(\varphi, \psi)(f_2(s)) = \psi(s)$ and $(\varphi, \psi)(s)\uparrow$ otherwise, $I = \lambda s.s$, $L = f_1$ and $R = f_2$. In other words, $\mathscr{S}$ is exactly the IOS of example 4.7, while $\mathscr{S}_0$ is the IOS of example 4.8.

Example 22.3. Example 21.2 with $E = \{\bot, \top\}$.

Proposition 22.2 (Example 22.4). Let $\mathscr{S}_0 = (\mathscr{F}_0, I, \Pi_0, L, R)$ be the IOS of example 22.3 and

$$\mathscr{F} = \{\varphi / \forall sxtr(\varphi(s,x,t), \varphi(s,x,r) = \top \Rightarrow r = t)\}.$$

Then $\mathscr{S} = (\mathscr{F}, I, \Pi_0 \upharpoonright \mathscr{F}^2, L, R)$ is a $(**)_0$, $(***)_0$-complete subspace of $\mathscr{S}_0$.

The proof repeats that of 22.1.

Members of $\mathscr{F}_0$ will be regarded as relations over $M \times N \times M$ or multiple-valued functions from $M \times N$ to M, writing $(s, x, t) \in \varphi$ or $t \in \varphi(s, x)$ for $\varphi(s, x, t) = \top$ etc. Therefore, $\mathscr{F} = \{\varphi / \varphi : M \times N \dashrightarrow M\}$, $\varphi \leq \psi$ iff $\varphi \subseteq \psi$, $\varphi\psi = \lambda sx.\psi(\varphi(s, x), x)$, $(\varphi, \psi)(f_1(s), x) = \varphi(s, x)$, $(\varphi, \psi)(f_2(s), x) = \psi(s, x)$ and $(\varphi, \psi)(s, x)\uparrow$ otherwise, $I = \lambda sx.s$, $L = \lambda sx.f_1(s)$ and $R = \lambda sx.f_2(s)$.

It is worth mentioning that example 22.2 can be obtained from example 21.4 by taking $E = \{\bot, \top\}$; similarly for example 22.4.

Example 22.5. Example 3.1, that is example 22.2 with $M = \omega$, $f_1 = \lambda s.2s$ and $f_2 = \lambda s.2s + 1$.

Notice that for each s there are unique k, l such that $s = 2^k(2l + 1) - 1 = f_2^k(f_1(l)) = \bar{k}(l)$. In other words, $\omega = \bigcup_n f_2^n(f_1(\omega))$, which implies $\langle I \rangle = (L, R) = I$ by exercise 21.1.

Proposition 21.2 provides the following explicit characterizations of the IOS-operations of example 22.5. Given a number s, then $s = \bar{k}(l)$ and

$$\langle \varphi \rangle(s) = f_2^k(f_1(\varphi(l))),$$
$$\Delta(\varphi, \psi)(s) = \psi^k(\varphi(l)),$$

while $[\varphi](s) = t$ iff there are $n, r_0, \ldots, r_n$ such that $r_0 = s$, $r_0, \ldots, r_{n-1}$ are odd, $r_{i+1} = \varphi((r_i - 1)/2)$ for all $i < n$, and $r_n = 2t$. That is,

$$[\varphi](s) = f_1^{-1}((f_2^{-1}\varphi)^n(s)),$$

where n is the least number m (if any) such that $(f_2^{-1}\varphi)^m(s)$ is even.

'Primitive recursive$_O$' will stand below for 'primitive recursive in the sense of Ordinary Recursion Theory' as opposed to the IOS-notion of primitive recursiveness.

The notions of primitive recursiveness$_O$ and μ-recursiveness for unary number theoretic functions turn out to be particular instances of primitive recursiveness and recursiveness in the space considered.

Proposition 22.3 (μ-Recursiveness Theorem). Let $\mathscr{S}$ be the IOS of example 22.5, $Z = \lambda s.R(s)\mathrm{sg}s$ and $Z_0 = \lambda s.2^s - 1$. Then the following equivalences hold for all $\varphi \in \mathscr{F}$, $\mathscr{B} \subseteq \mathscr{F}$.

(1) φ is primitive recursive$_O$ (μ-recursive) in $\mathscr{B}$ iff φ is primitive recursive (recursive) in $\{Z_0\} \cup \mathscr{B}$.

(2) φ is μ-recursive in $\mathscr{B}$ iff φ is recursive in $\{Z\} \cup \mathscr{B}$.

Proof. The 'if'-parts of (1), (2) are immediate since L, R, Z, Z_0 are primitive recursive$_O$ functions and $\circ, \Pi, \Delta$ are primitive recursive$_O$ operations, while $[\ \ ]$ is a μ-recursive operation by the explicit characterizations of Δ, $[\ \ ]$.

Let us show that for every $\varphi \in \mathscr{F}$ there is a unique $\tilde{\varphi} \in \mathscr{F}$ representing φ in the sense of chapter 8. Take $\tilde{\varphi}(s) = \tilde{\varphi}(\bar{k}(l)) = \overline{\varphi(k)}(l)$ by definition. Then $\bar{k}\tilde{\varphi} = \overline{\varphi(k)}$ for all k; hence $\tilde{\varphi}$ represents φ. Supposing that φ_1 also represents φ, we get $\varphi_1(s) = \bar{k}\varphi_1(l) = \bar{k}\tilde{\varphi}(l) = \tilde{\varphi}(s)$ for all s; hence $\varphi_1 = \tilde{\varphi}$.

If φ is primitive recursive$_O$ (μ-recursive) in $\mathscr{B}$, then 8.1 (respectively, 8.3***) implies that $\tilde{\varphi}$ is primitive recursive (recursive) in $\mathscr{B}^{\sim}$, hence it suffices to show that φ is primitive recursive in $Z_0, \tilde{\varphi}$, $\tilde{\psi}$ is primitive recursive in Z_0, ψ and Z_0 is recursive in Z.

Take $\psi_0 = \Delta(R, L)$, $\psi_1 = \mu\theta.(Z(I, \theta R), L)$ and $\psi_2 = \mu\theta.Z(I, \psi_1\theta R)$.

An easy induction on s implies that $\psi_0 = \lambda s.s+1$. If $s = 2l$, then

$$\psi_0(s) = \bar{0}\psi_0(l) = R(l) = 2l+1 = s+1.$$

If $s = 2l+1$, then

$$\psi_0(s) = R\psi_0(l) = \psi_0 L(l) = 2(l+1) = s+1.$$

It also follows by induction that $\psi_1 = \lambda s.s \dot{-} 1$. First,

$$\psi_1(0) = Z(I, \psi_1 R)(0) = (I, \psi_1 R)(0) = 0.$$

If $s = 2l+1$, then

$$\psi_1(s) = (Z(I, \psi_1 R), L)(2l+1) = L(l) = s-1.$$

If $s = 2l+2$, then

$$\psi_1(s) = Z(I, \psi_1 R)(l+1) = (I, \psi_1 R)(2(l+1)+1) = \psi_1 R(l+1) = R(l) = s-1.$$

We get

$$\begin{aligned}\psi_2(0) &= Z(I, \psi_1\psi_2 R)(0) = I(0) = 0,\\ \psi_2(s+1) &= (I, \psi_1\psi_2 R)(2(s+1)+1)\\ &= \psi_1\psi_2 R(s+1) = \psi_2 R(s) = 2\psi_2(s)+1,\end{aligned}$$

hence $\psi_2 = \lambda s.R^s(0) = \lambda s.\bar{s}(0) = Z_0$. The elements ψ_1, ψ_2 are recursive in Z by the proof of 6.39, hence so is Z_0.

Take $\psi_3 = \Delta(I, \psi_0)$ and

$$\rho = \langle Z_0 \rangle G \langle \psi_3 \psi Z_0 \rangle G \langle \psi_3 \rangle.$$

If $\psi(k) = t$, then

$$\begin{aligned}\bar{k}\rho(l) &= Z_0\langle \bar{k}\rangle\langle \psi_3\psi Z_0\rangle G\langle\psi_3\rangle(l) = Z_0\langle \psi_0^k\psi Z_0\rangle G\langle\psi_3\rangle(l)\\ &= \bar{l}\langle \psi_0^k\psi Z_0\rangle G\langle\psi_3\rangle(0) = \psi_0^k\psi Z_0\langle \bar{l}\psi_3\rangle(0) = \psi Z_0\langle\psi_0^l\rangle(k)\\ &= Z_0\langle\psi_0^l\rangle(t) = \bar{t}\langle\psi_0^l\rangle(0) = \psi_0^l t(0) = \bar{t}(l)\end{aligned}$$

for all l, hence $\bar{k}\rho = \bar{t}$. If $\psi(k)\uparrow$, then similarly $\bar{k}\rho(l) = \psi Z_0\langle\psi_0^l\rangle(k)\uparrow$ for all l, hence $\bar{k}\rho = O$. Therefore, $\tilde{\psi} = \rho$; hence $\tilde{\psi}$ is primitive recursive in Z_0, ψ.

Finally, if $\varphi(k) = t$, then $\bar{k}\tilde{\varphi} = \bar{t}$; hence

$$Z_0\tilde{\varphi}\psi_3(k) = \bar{k}\tilde{\varphi}\psi_3(0) = \bar{t}\psi_3(0) = \psi_0^t(0) = t.$$

If $\varphi(k)\uparrow$, then $\bar{k}\tilde{\varphi} = O$; hence

$$Z_0\tilde{\varphi}\psi_3(k) = \bar{k}\tilde{\varphi}\psi_3(0) = O\psi_3(0)\uparrow.$$

We get $\varphi = Z_0\tilde{\varphi}\psi_3$, hence φ is primitive recursive in $Z_0, \tilde{\varphi}$. The proof is complete.

It was necessary to add Z or Z_0 to the initial elements for, as will be shown in the exercises, neither of them is IOS-recursive.

Proposition 22.3 yields a natural way of generating the unary partial recursive functions; other unary bases are provided by 7.11 and exercise 7.3. For instance, φ is partial recursive iff

$$\varphi \in cl(L, \langle\langle L\rangle\rangle, \langle\langle A\rangle\rangle, \langle Z\rangle/\circ, [\ \ ]).$$

The following analogue to 22.3 for mappings is obtained by an immediate parametrization.

Proposition 22.4. Let $\mathscr{S}$ be the space of example 22.5, $\mathscr{B} \subseteq \mathscr{F}$ and $\Gamma:\mathscr{F}^n \to \mathscr{F}$, $n > 0$. Then the following equivalences hold.

(1) Γ is primitive recursive$_0$ (μ-recursive) in $\mathscr{B}$ iff Γ is primitive recursive (recursive) in $\{Z_0\} \cup \mathscr{B}$.

(2) Γ is μ-recursive in $\mathscr{B}$ iff Γ is recursive in $\{Z\} \cup \mathscr{B}$.

The following example corresponds to example 8 in Skordev [1980], chapter 2.

Example 22.6. Example 22.1 with set M and splitting scheme as in example 22.5. Therefore, $\mathscr{F} = \{\varphi/\varphi \subseteq \omega^2\} = \{\varphi/\varphi:\omega \to 2^\omega\}$, $\varphi \le \psi$ iff $\varphi \subseteq \psi$ iff $\forall s(\varphi(s) \subseteq \psi(s))$, $\varphi\psi = \lambda s.\psi(\varphi(s)) = \lambda s.\cup\{\psi(r)/r \in \varphi(s)\}$,

$$(\varphi, \psi)(s) = \begin{cases} \varphi(s/2), & \text{if } s \text{ is even,} \\ \psi((s-1)/2), & \text{if } s \text{ is odd,} \end{cases}$$

$I = \lambda s.s$, $L = f_1$ and $R = f_2$.

The explicit characterizations of 21.2 now read as follows.

$$\langle \varphi \rangle(s) = f_2^k(f_1(\varphi(l))),$$
$$\Delta(\varphi, \psi)(s) = \psi^k(\varphi(l)),$$

where $s = \overline{k}(l)$,

$t \in [\varphi](s)$ iff there are $n, r_0, \ldots, r_n$ such that $r_0 = s, r_0, \ldots, r_{n-1}$ are odd, $r_{i+1} \in \varphi((r_i - 1)/2)$ for all $i < n$, r_n is even and $t = r_n/2$.

The following two Partial Recursiveness Theorems embody the notions of relative partial recursiveness for single-valued and multiple-valued functions. The latter is introduced by the definition given in the remarks to exercise 8.2, allowing f to be multiple-valued; alternatively, adding to the definition of relative μ-recursiveness a certain multiple-valued initial function, say $\lambda s.\{0, 1\}$ or $\lambda s.\omega$. We recall that the space of example 22.6 has an element satisfying the assumptions of exercise 7.9***, $U = \lambda s.\{2s, 2s+1\}$.

Proposition 22.5. Let $\mathscr{S}$ be the IOS of example 22.6 and $\mathscr{S}_1$ its subspace consisting of all the single-valued functions. (In other words, $\mathscr{S}_1$ is the space of example 22.5.) Let $\varphi \in \mathscr{F}_1$, $\mathscr{B} \subseteq \mathscr{F}_1$. Then

(1) φ is partial recursive in $\mathscr{B}$ iff φ is recursive in $\{Z, U\} \cup \mathscr{B}$.

Remark. Though $\varphi \in \mathscr{F}_1$ and $\mathscr{B} \subseteq \mathscr{F}_1$, the considerations could not be carried out within $\mathscr{S}_1$ since $U \in \mathscr{F} \setminus \mathscr{F}_1$.

Proof. It is the 'only-if'-part of (1) that needs proof.

As shown in the proof of 22.3, there is a $\tilde{\varphi} \in \mathscr{F}_1$ representing φ. If $\varphi_1 \in \mathscr{F}$ also represents φ, it follows as in the proof of 22.3 that $\varphi_1 = \tilde{\varphi}$. Moreover, φ is recursive in $Z, \tilde{\varphi}$ and $\tilde{\psi}$ is recursive in Z, ψ by the proof of 22.3.

If φ is partial recursive in $\mathscr{B}$, then $\tilde{\varphi}$ is recursive in $\{U\} \cup \mathscr{B}^{\sim}$ by exercise 8.2***, hence φ is recursive in $\{Z, U\} \cup \mathscr{B}$. The proof is complete.

Proposition 22.6. Let $\mathscr{S}$ be the space of example 22.6, $\varphi \in \mathscr{F}$ and $\mathscr{B} \subseteq \mathscr{F}$. Then the following equivalences hold:

(1) φ is partial recursive in $\mathscr{B}$ iff φ is recursive in $\{Z, U\} \cup \mathscr{B}$.

(2) φ is a recursively enumerable relation iff φ is recursive in Z, U.

Proof. To each $\varphi \in \mathscr{F}$ assign a single-valued φ^* such that $\varphi^*(\bar{k}(l)) = k$, if $k \in \varphi(l)$, and $\varphi^*(\bar{k}(l))\uparrow$ otherwise. Then it follows that $\varphi^* = \langle \varphi \rangle \psi_0$, where $\psi_0(\bar{k}(k)) = k$ and $\psi_0(s)\uparrow$ otherwise. The element ψ_0 is a partial recursive function, hence it is recursive in Z by 22.3. On the other hand, we get $\varphi = \sup_k \bar{k}\varphi^* = \bar{\omega}\varphi^*$, where $\bar{\omega}$ is the element considered in exercise 7.9***. Therefore, φ^* is recursive in Z, φ, while φ is recursive in U, φ^*. The equivalence (1) is deduced as follows.

$$\begin{array}{lll} & \varphi \text{ is partial recursive in } \mathscr{B} & \\ \text{iff} & \varphi^* \text{ is partial recursive in } \mathscr{B}^* & \\ \text{iff} & \varphi^* \text{ is recursive in } \{Z, U\} \cup \mathscr{B}^* & \text{(by 22.5)} \\ \text{iff} & \varphi \text{ is recursive in } \{Z, U\} \cup \mathscr{B}. & \end{array}$$

Since a relation is *recursively enumerable* iff it is partial recursive as a multiple-valued function, (2) follows immediately by taking $\mathscr{B} = \varnothing$. This completes the proof.

Example 22.7. Example 22.4 with $N = M^{n-1}$, $n > 0$. (It is a subspace of example 21.5 as well.)

Example 22.8. Example 22.7 with set M and splitting scheme as in example 22.5. Thus $\mathscr{F} = \{\varphi / \varphi : \omega^n \dashrightarrow \omega\}$, $\varphi \leq \psi$ iff $\varphi \subseteq \psi$, $\varphi\psi = \lambda s_1 \ldots s_n . \psi(\varphi(s_1, \ldots, s_n), s_2, \ldots, s_n)$,

$$(\varphi, \psi)(s_1, \ldots, s_n) = \begin{cases} \varphi(s_1/2, s_2, \ldots, s_n), & \text{if } s_1 \text{ is even,} \\ \psi(((s_1 - 1)/2), s_2, \ldots, s_n), & \text{if } s_1 \text{ is odd,} \end{cases}$$

$I = \lambda s_1 \ldots s_n . s_1$, $L = \lambda s_1 \ldots s_n . 2s_1$ and $R = \lambda s_1 \ldots s_n . 2s_1 + 1$.

Then for each s_1 there are unique k, l such that

$$s_1 = f_2^k(f_1(l)) = \bar{k}(l, s_2, \ldots, s_n)$$

for all $s_2, \ldots, s_n$, hence there are unique $k_1, \ldots, k_n, l$ such that $s_1 = \bar{k}_1 \ldots \bar{k}_n(l, s_2, \ldots, s_n)$ for all $s_2, \ldots, s_n$. This implies that for every $\varphi \in \mathscr{F}$ there is a unique $\tilde{\varphi} \in \mathscr{F}$ representing it in the sense of chapter 8, namely

$$\tilde{\varphi}(\bar{k}_1 \ldots \bar{k}_n(l, s_2, \ldots, s_n), s_2, \ldots, s_n) = \overline{\varphi(k_1, \ldots, k_n)}(l, s_2, \ldots, s_n).$$

Proposition 21.4 gives the following explicit characterizations of $\langle \ \rangle$, Δ, $[\ ]$, which show that the first two operations are primitive recursive$_0$, while the last one is μ-recursive.

$$\langle \varphi \rangle (s_1, \ldots, s_n) = f_2^k(f_1(\varphi(l, s_2, \ldots, s_n))),$$

$$\Delta(\varphi, \psi)(s_1, \ldots, s_n) = \varphi\psi^k(l, s_2, \ldots, s_n),$$

where $s_1 = f_2^k(f_1(l))$,
$[\varphi](s_1, \ldots, s_n) = t$ iff there are $n, r_0, \ldots, r_n$ such that $r_0 = s_1, r_0, \ldots, r_{n-1}$ are odd, $r_{i+1} = \varphi((r_i - 1)/2, s_2, \ldots, s_n)$ for all $i < n$, r_n is even and $t = r_n/2$.

The following statement describes the primitive recursiveness$_O$ and μ-recursiveness for n-ary number functions in terms of the corresponding IOS-notions. Other characterizations may be obtained by making use of 7.11 and exercise 7.3.

Proposition 22.7. Let $\mathscr{S}$ be the IOS of example 22.8, $Z = \lambda s_1 \ldots s_n . f_2(s_1)\mathrm{sg} s_1$, $Z_0 = \lambda s_1 \ldots s_n . 2^{s_1} - 1$ and $I_i^n = \lambda s_1 \ldots s_n . s_i$, $1 < i \leq n$. Then the following equivalences hold for all $\varphi \in \mathscr{F}$, $\mathscr{B} \subseteq \mathscr{F}$.

(1) φ is primitive recursive$_O$ (μ-recursive) in $\mathscr{B}$ iff φ is primitive recursive (recursive) in $\{Z_0, I_2^n, \ldots, I_n^n\} \cup \mathscr{B}^{\sim}$.

(2) φ is μ-recursive in $\mathscr{B}$ iff φ is recursive in $\{Z, I_2^n, \ldots, I_n^n\} \cup \mathscr{B}^{\sim}$.

In particular,

(3) φ is primitive recursive$_O$ (partial recursive) iff φ is primitive recursive (recursive) in $Z_0, I_2^n, \ldots, I_n^n$.

(4) φ is partial recursive iff φ is recursive in $Z, I_2^n, \ldots, I_n^n$.

Proof. The 'if'-parts of (1), (2) are immediate. In order to establish their 'only if'-parts it suffices to show that Z_0 is recursive in Z and φ is primitive recursive in $Z_0, I_2^n, \ldots, I_n^n, \tilde{\varphi}$. This completes the proof by making use of 8.1, 8.3***.

Proposition 21.5 implies that the space of example 22.5 is isomorphic to a subspace of the present IOS, where the isomorphism assigns to each unary function f the n-ary one $\lambda s_1 \ldots s_n . f(s_1)$. Therefore, Z_0 is recursive in Z by the proof of 22.3.

Take ψ_0, ψ_3 to correspond to the elements ψ_0, ψ_3 of the proof of 22.3. Then ψ_0, ψ_3 are primitive recursive in Z_0 and $\psi_0 = \lambda s_1 \ldots s_n . s_1 + 1$, $\bar{k}\psi_3 = \psi_0^k$.

Let G_{n-1} be the element constructed in 7.15 and

$$\psi_4 = Z_0 \langle I_n^n Z_0 \langle I_{n-1}^n Z_0 \ldots \langle I_2^n Z_0 \rangle \ldots \rangle\rangle G_{n-1}.$$

Then ψ_4 is primitive recursive in $Z_0, I_2^n, \ldots, I_n^n$ and

$$\psi_4(s_1, \ldots, s_n) = \overline{s_1} \ldots \overline{s_n}(0, s_2, \ldots, s_n).$$

If $\varphi(s_1, \ldots, s_n) = t$, then $\overline{s_1} \ldots \overline{s_n}\tilde{\varphi} = \bar{t}$, hence

$$\begin{aligned}\psi_4\tilde{\varphi}\psi_3(s_1, \ldots, s_n) &= \overline{s_1} \ldots \overline{s_n}\tilde{\varphi}\psi_3(0, s_2, \ldots, s_n) = \bar{t}\psi_3(0, s_2, \ldots, s_n) \\ &= \psi_0^t(0, s_2, \ldots, s_n) = t.\end{aligned}$$

If $\varphi(s_1, \ldots, s_n)\uparrow$, then $\overline{s_1} \ldots, \overline{s_n}\tilde{\varphi} = O$, hence

$$\psi_4\tilde{\varphi}\psi_3(s_1, \ldots, s_n) = \overline{s_1} \ldots \overline{s_n}\tilde{\varphi}\psi_3(0, s_2, \ldots, s_n) = O\psi_3(0, s_2, \ldots, s_n)\uparrow.$$

We get $\varphi = \psi_4\tilde{\varphi}\psi_3$; hence φ is primitive recursive in $Z_0, I_2^n, \ldots, I_n^n$, $\tilde{\varphi}$. The proof is complete.

The reader has undoubtedly noticed the peculiarities which distinguish (1), (2) from the corresponding equivalences of 22.3. We did not prove that φ was μ-recursive in $\mathscr{B}$ iff it was recursive in $\{Z, I_2^n, \ldots, I_n^n\} \cup \mathscr{B}$ because of our inability to work out full compositions unless $n = 1$: all the arguments of a function except the first one were parameters taken into account but not affected. This obstacle will be overcome in the exercises below either by making use of a t-operation or by considering the space of functions $\varphi:\omega^n \dashrightarrow \omega^n$ instead.

The following wider space consists of functions operating on arbitrary tuples of natural numbers.

Example 22.9. The IOS $\mathscr{S}$ of example 22.2 with $M=\omega^*=\bigcup_n\omega^n$, $f_1=\lambda x.(0,x)$, $f_2(\Lambda)=\Lambda$ and $f_2(s,x)=(s+1,x)$.

If $x=(s_1,\ldots,s_n)$, then we write (s,x) for $(s,s_1,\ldots,s_n)$; Λ is the empty tuple, the only member of ω^0.

It follows that $\bar{s_1}\ldots\bar{s_n}(x)=(s_n,\ldots,s_1,x)$ for all $x\in\omega^*$. In particular, $\bar{s}(x)=(s,x)$ and $\bar{s}(\Lambda)=s$. We get $(\varphi,\psi)(0,x)=\varphi(x)$, $(\varphi,\psi)(\Lambda)=\psi(\Lambda)$ and $(\varphi,\psi)(s+1,x)=\psi(s,x)$. Notice that $f_1(\omega^*)\cup f_2(\omega^*)=\omega^*$ but $\bigcup_n f_2^n(f_1(\omega^*))\neq\omega^*$, hence $(L,R)=I$, $\langle I\rangle\neq I$ by exercise 21.1. Proposition 21.2 gives $\langle\varphi\rangle(\Lambda)\uparrow$ and

$$\langle\varphi\rangle(s,x)=\bar{s}\langle\varphi\rangle(x)=\varphi\bar{s}(x)=(s,\varphi(x)),$$

while $[\varphi](x)=y$ iff there are $n,z_0,\ldots,z_n$ such that $z_0=x$, $z_i\in f_2(\omega^*)$ and $z_{i+1}=\varphi(f_2^{-1}(z_i))$ for all $i<n$, and $z_n=(0,y)$.

The following Stack Recursiveness Theorem formulates the notion of partial recursive stack function introduced in Germano and Maggiolo-Schettini [1976] in terms of operations close to the basic IOS-operations in this example.

Proposition 22.8. Let $\mathscr{S}$ be the IOS of example 22.9, $Z(\Lambda)=0$ and $Z(s,x)=(s+1,x)$. Then $\varphi\in\mathscr{F}$ is a partial recursive stack function iff φ is recursive in Z.

Proof. The *partial recursive stack functions* are generated from the initial ones $\mathbb{O},\mathbb{S},\mathbb{P}$ by means of multiplication, which is exactly the semigroup multiplication of $\mathscr{F}$, left cylindrification operator $\lambda\varphi.{}^c\varphi$ and repetition operator $\lambda\varphi.\varphi^\nabla$, where $\mathbb{O}=L$, $\mathbb{S}=Z$, $\mathbb{P}(\Lambda)=\Lambda$ and $\mathbb{P}(s,x)=(s\dot{-}1,x)$, ${}^c\varphi(\Lambda)=0$, ${}^c\varphi(s,x)=(s,\varphi(x))$, $\varphi^\nabla(x)=y$ iff there are $n,z_0,\ldots,z_n$ such that $z_0=x$, $z_i\in f_2(\omega^*)$ and $z_{i+1}=\varphi(z_i)$ for all $i<n$, and $z_n=(0,y)$.

It follows almost immediately that $\mathbb{P}=(L,I)$, ${}^c\varphi=Z(L,\langle\varphi\rangle)$ and $\varphi^\nabla=[R\varphi]$; hence all partial recursive stack functions are recursive in Z.

On the other hand, $R=\mathbb{S}\mathbb{O}^\nabla$. Notice that $R\mathbb{P}=I$ and $\mathbb{O}\varphi^\nabla=I$ for all φ.

Writing $\sigma_0,\sigma_1,\sigma_2,\sigma$ respectively for $({}^c\mathbb{P}\mathbb{P}\psi\mathbb{O}^2)^\nabla$, $(\mathbb{P}\varphi\mathbb{O})^\nabla$, $(\mathbb{P}^cR\sigma_0\sigma_1\mathbb{O}^2)^\nabla$, $\mathbb{S}\sigma_2(\psi\mathbb{O})^\nabla$, one gets

$$\begin{aligned}\sigma(0,x)&=\sigma_2(\psi\mathbb{O})^\nabla(1,x)={}^cR\sigma_0\sigma_1\mathbb{O}(\psi\mathbb{O})^\nabla(0,x)=\sigma_0\sigma_1(0,R(x))\\&=\sigma_1(R(x))=\varphi\mathbb{O}(\mathbb{P}\varphi\mathbb{O})^\nabla(x)=\varphi(x),\\\sigma(\Lambda)&=\sigma_2(\psi\mathbb{O})^\nabla(0)=(\psi\mathbb{O})^\nabla(\Lambda)=\psi(\Lambda),\\\sigma(s+1,x)&=\sigma_2(\psi\mathbb{O})^\nabla(s+2,x)\\&={}^cR\sigma_0\sigma_1\mathbb{O}(\psi\mathbb{O})^\nabla(s+1,x)=\sigma_0\sigma_1(s+1,R(x))\\&=\psi\mathbb{O}\sigma_1(s,x)=\psi(s,x);\end{aligned}$$

hence $(\varphi,\psi)=\sigma$. It also follows that $\langle\varphi\rangle=\mathbb{S}(\mathbb{P}^\nabla,{}^c\varphi)$ and $[\varphi]=(\mathbb{P}\varphi)^\nabla$, which completes the proof.

EXERCISES TO CHAPTER 22

Exercise 22.1. Let $\mathscr{S}$ be the space of example 22.5. Show that the members $Z, Z_0, \lambda s.s \dot{-} 1$ and $\psi_0 = \lambda s.0$ of $\mathscr{F}$ are not IOS-recursive.

Hint. It suffices to show that whenever φ is recursive, then either $\exists n(L^n\varphi \in \mathscr{D})$ or $\psi_0\varphi = O$, for the functions in question do not have this property. The element φ has a normal form $\bar{I}[\sigma]$ for a certain primitive σ. Notice that $\forall n(\psi_0 = \psi_0 L^n)$, while $\forall \alpha \in \mathscr{D} \exists n(L^n\alpha\sigma \in \mathscr{D})$ by the hint to exercise 6.12. Using the unwinding method, show that either $\exists n(L^n\varphi \in \mathscr{D})$ or $\psi_0\varphi = O$. The segment to be used is of the form $\{\theta/\forall n(\tau_n\theta \leq O)\}$; though not regular, it contains $[\sigma]$ whenever closed under $\lambda\theta.(I, \sigma\theta)$.

Exercise 22.2. Let $\mathscr{S}$ be an IOS and let $f: \omega^n \to \omega$ be represented by a primitive recursive element φ. Show that f is a primitive recursive function.

Hint. Assume $n = 1$. Let $\mathscr{S}_0$ be the space of example 22.5 and let φ_0 be a primitive recursive member of $\mathscr{F}_0$ the construction of which repeats that of φ. It follows as in the hint to exercise 6.13 that for all s there are m and $\alpha \in \mathscr{D}$ such that $L^m\bar{s}\varphi = \alpha$. As a consequence of the IOS-axioms, this holds both in $\mathscr{S}$ and $\mathscr{S}_0$; hence $L^m\bar{s}\varphi_0 = \alpha$ as well. However, $\bar{s}\varphi = \overline{f(s)}$; hence $\alpha = L^m\overline{f(s)}$. Therefore, $\bar{s}\varphi_0(0) = \overline{f(s)}(0)$ for all s, hence $f = Z_0\varphi_0\psi_3$ with Z_0, ψ_3 as in the proof of 22.3.

Exercise 22.3. Let $\mathscr{S}$ be the space of example 22.8 and let

$$St(\varphi)(\bar{t}_1 \ldots \bar{t}_n(s_1), s_2, \ldots, s_n) = \overline{\varphi(t_1, \ldots, t_n)}\,\bar{t}_2 \ldots \bar{t}_n(s_1),$$

$\bar{k}(s)$ standing for $f_2^k(f_1(s))$. Show that St is a t-operation with a set of functionary elements $\mathscr{B}_0 = \{Z_0, I_2^n, \ldots, I_n^n\}$ or $\mathscr{B}_0 = \{Z, I_2^n, \ldots, I_n^n\}$ and the following equivalences hold for all $\varphi \in \mathscr{F}$, $\mathscr{B} \subseteq \mathscr{F}$.

(1) φ is primitive recursive$_0$ (μ-recursive) in $\mathscr{B}$ iff φ is primitive st-recursive (st-recursive) in $\mathscr{B}$, provided we take the former $\mathscr{B}_0$.

(2) φ is μ-recursive in $\mathscr{B}$ iff φ is st-recursive in $\mathscr{B}$, provided we take the latter $\mathscr{B}_0$.

Hint. Show by exercise 21.1 that St is a storing operation. Construct elements K_3, K_4 primitive recursive in Z_0, $I_2^n, \ldots, I_n^n$ such that $\varphi = K_3 St(\varphi) K_4$ for all φ. Make use of 22.7 and the equality $\tilde{\psi} = St(\psi)\Delta(I, I)^{n-1}$.

Example 22.10. Example 22.2 with $M = \omega^n, n > 0, f_1 = \lambda s_1 \ldots s_n.(2s_1, s_2, \ldots, s_n)$ and similarly for f_2.

Consider the elements $Z_0 = \lambda s_1 \ldots s_n.(2^{s_1} - 1, s_2, \ldots, s_n)$, $Z = \lambda s_1 \ldots s_n.((2s_1 + 1)\mathrm{sg}s_1, s_2, \ldots, s_n)$ and $I_{n,i} = \lambda s_1 \ldots s_n.(s_1', \ldots, s_n')$, $1 < i \leq n$, where $s_1' = s_i$, $s_i' = s_1$ and $s_j' = s_j$ otherwise. For all $\varphi \in \mathscr{F}$, $\mathscr{B} \subseteq \mathscr{F}$ let $\varphi^{(i)}$ stand for the i-th co-ordinate function of φ and $\mathscr{B}^{(i)}$ for the set of i-th co-ordinate functions of members of $\mathscr{B}$.

Exercise 22.4. Let $\mathscr{S}$ be the IOS of example 22.10. Prove that the following equivalences hold.

(1) φ is primitive recursive (recursive) in $\{Z_0, I_{n,2}, \ldots, I_{n,n}\} \cup \mathscr{B}$ iff $\varphi^{(1)}, \ldots, \varphi^{(n)}$ are primitive recursive_O (μ-recursive) in $\bigcup_{i=1}^{n} \mathscr{B}^{(i)}$.

(2) φ is recursive in $\{Z, I_{n,2}, \ldots, I_{n,n}\} \cup \mathscr{B}$ iff $\varphi^{(1)}, \ldots, \varphi^{(n)}$ are μ-recursive in $\bigcup_{i=1}^{n} \mathscr{B}^{(i)}$.

Hint. Reduce (1), (2) to the corresponding equivalences of 22.3.

Exercise 22.5. Let φ be a n-ary number function and $\mathscr{B}$ a set of such functions, $\varphi^* = \lambda s_1 \ldots s_n.(\varphi(s_1, \ldots, s_n), s_2, \ldots, s_n)$ and let $\mathscr{S}$ be the IOS of example 22.10 so that $\varphi^* \in \mathscr{F}$, $\mathscr{B}^* \subseteq \mathscr{F}$. Prove the following equivalences.

(1) φ is primitive recursive_O (μ-recursive) in $\mathscr{B}$ iff φ^* is primitive recursive (recursive) in $\{Z_0, I_{n,2}, \ldots, I_{n,n}\} \cup \mathscr{B}^*$.

(2) φ is μ-recursive in $\mathscr{B}$ iff φ^* is recursive in $\{Z, I_{n,2}, \ldots, I_{n,n}\} \cup \mathscr{B}^*$.

Example 22.11. Example 22.2 with $M = \{\ldots \varepsilon_1 \varepsilon_0 * \delta_0 \delta_1 \ldots / \forall n(\varepsilon_n, \delta_n \in \{0,1\})\}$, $f_1 = \lambda x * y.x0 * y$ and $f_2 = \lambda x * y.x1 * y$.

Members of M can be regarded as records on an infinite tape, the asterisk indicating the scanned symbol. Let $S_1 = \lambda x\varepsilon * y.x * \varepsilon y$ and $S_2 = S_1^{-1}$. An element $\psi \in \mathscr{F}$ is *normal* iff

$$\forall x y x' y' (\psi(x * y) = x' * y' \Rightarrow \forall y'' (\psi(x * y'') = x' * y''))$$

Exercise 22.6. Let $\mathscr{S}$ be the IOS of example 22.11 and let $\mathscr{B}$ consist of normal elements. Show that φ is recursive in $\{S_1, S_2\} \cup \mathscr{B}$ iff φ is prime recursive in $\{S_1, S_2\} \cup \mathscr{B}$.

Hint. Take $W = S_2$, $W_1 = LS_1$, $W_2 = RS_1$ and use 21.10.

CHAPTER 23

Functions computable by programs

While the previous chapter was devoted to number functions and relations, the present chapter (as well as the next) deals with the more general case of functions on arbitrary domains. Concepts of computability for such functions are provided in particular by certain kinds of abstract programs studied in Computer Science, e.g. program schemes augmented by counters and stacks, which correspond to the so-called finite algorithmic procedures with counting (Friedman [1971]) and stacking (Moldestad, Stoltenberg-Hansen and Tucker [1981]). The considerations below aim to show that those notions of computability are particular instances of relative IOS-recursiveness, with all the attendant consequences.

Let us start with the simplest program schemes, the unary ones. All the other kinds will be obtained in terms of these.

We have unary function symbols $f, f_1, f_2, \ldots$ and unary predicate symbols $P, P_1, P_2, \ldots$. A *unary scheme* consists of instructions of the form

$$l_i \text{ do } f \text{ go to } l_j,$$
$$l_i \text{ if } P \text{ then go to } l_j \text{ else go to } l_k$$

and one instruction l_m stop. We shall write simply $l_i f l_j$ and l_i if P then l_j else l_k. Each label l_i occurs just once as a prefix of an instruction; the instruction itself will often be referred to as l_i. Therefore, a scheme is a nonempty list $l_1, \ldots, l_m$ of instructions one of which, say l_1, is declared to be first.

Unary schemes are interpreted by means of unary bases to give interpreted schemes or *programs* in those bases. A *unary basis* is a tuple

$$B = (M, f_1, \ldots, f_u, P_1, \ldots, P_v),$$

where M is a nonempty set, $f_i: M \longrightarrow M$, $0 \le i \le u$, and $P_i: M \longrightarrow \{0, 1\}$, $0 \le i \le v$. Function and predicate symbols are interpreted respectively by functions and predicates from B. The same letters will denote function symbols and functions, respectively predicate symbols and predicates.

The execution of a program $\mathscr{P}$ employs one processing *register* r and goes in a fairly ordinary manner. It starts with the first instruction; $l_i f l_j$ replaces r by $f(r)$ and the computation continues with l_j, provided $f(r)\downarrow$; in the case of l_i if P then l_j else l_k the content of the register remains unaltered and the next instruction to be executed is either l_j or l_k, depending on whether $P(r) = 0$ or

$P(r)=1$. If $f(r)\uparrow$ or $P(r)\uparrow$, then the computation does not terminate; another instance in which the computation process fails to terminate is that of an infinite computation. If the last instruction l_m stop is reached, then the process terminates and the content of r is the final result. Therefore, the program $\mathscr{P}$ computes a partial function $\varphi_{\mathscr{P}}: M \dashrightarrow M$.

If l_2 is taken as a first instruction, then another function φ_2 will be computed etc. If l_m is the first instruction, then the function computed will, of course, be $\varphi_m = I = \lambda r.r$.

Each program $\mathscr{P}$ has a corresponding *characteristic system*

(1) $$\Gamma_i(\theta_1, \ldots, \theta_m) = \theta_i, \quad 1 \le i \le m,$$

the equalities of which are obtained directly from the instruction of $\mathscr{P}$. Namely, $l_i f l_j$ yields the equality $f\theta_j = \theta_i$, l_i if P then l_j else l_k yields $(P \to \theta_j, \theta_k) = \theta_i$ and l_m stop yields $I = \theta_m$, where $\varphi\psi = \lambda r.\psi(\varphi(r))$ and

$$(P \to \varphi, \psi)(r) = \begin{cases} \varphi(r), & \text{if } P(r) = 0, \\ \psi(r), & \text{if } P(r) = 1, \\ \uparrow, & \text{if } P(r)\uparrow. \end{cases}$$

We shall make use of the fact that $\varphi_{\mathscr{P}}, \varphi_2, \ldots, \varphi_m$ is the least solution of (1) (Mazurkievicz [1971]; cf. exercise 23.1). Alternatively, one may formally define the execution of a program $\mathscr{P}$ as the first component of the least solution to its characteristic system. (Why does such a solution exist?) Of course, the equalities of (1) can be replaced by inequalities.

A few specific classes of program schemes will be considered now. These are unary schemes with several interpreted function and predicate symbols, i.e., function and predicate constants. Correspondingly, the presence of such constants assumes interpretations by means of more specific unary bases.

Firstly, there are unary schemes with *counters*, which we shall call *C-schemes*. It suffices to have two counting registers since the work of a greater number of counters can be modelled by just two.

A *unary C-scheme* is a unary scheme in which some of the function constants L, R, K, L_2, R_2, K_2 and the predicate constants Ev, Ze, Ev_2, Ze_2 may eventually occur.

Given a unary basis $B = (M, f_1, \ldots, f_u, P_1, \ldots, P_v)$, C-schemes are interpreted by means of the modified unary basis

$$B^c = (\omega^2 \times M, f_1^c, \ldots, f_u^c, P_1^c, \ldots, P_v^c),$$

where

$$f_i^c = \lambda c_1 c_2 r.(c_1, c_2, f_i(r)), i = 1, \ldots, u,$$
$$P_i^c = \lambda c_1 c_2 r.P_i(r), i = 1, \ldots, v.$$

The interpretation of the constants does not depend on B:

$$L = \lambda c_1 c_2 r.(2c_1, c_2, r),$$
$$R = \lambda c_1 c_2 r.(2c_1 + 1, c_2, r),$$
$$K = \lambda c_1 c_2 r.([c_1/2], c_2, r),$$
$$Ev = \lambda c_1 c_2 r.\mathrm{rem}(c_1, 2),$$
$$Ze = \lambda c_1 c_2 r.\mathrm{sg}c_1$$

and similarly $L_2, R_2, K_2, Ev_2, Ze_2$ operate the second counter c_2. The predicates Ze, Ze_2 are not really needed for computations involving functions over M. They are added to ensure that all partial recursive functions can be computed in the counters, including the successor and predecessor functions traditionally used to operate counters. (Cf. 22.3 and exercise 22.1.)

The interpreted C-schemes are C-*programs* in the basis B. Such a program $\mathscr{P}$ computes a function $\varphi_{\mathscr{P}}:\omega^2 \times M \dashrightarrow \omega^2 \times M$. (Usually the first counter c_1 or the register r are regarded as outputs,) A function $\varphi:\omega^2 \times M \dashrightarrow \omega^2 \times M$ is said to be C-*computable in* B iff $\varphi = \varphi_{\mathscr{P}}$ for a certain C-program $\mathscr{P}$ in B.

In order to characterize C-computability we design an appropriate space.

Example 23.1. The IOS $\mathscr{S}$ of example 22.2 with $\omega^2 \times M, L, R$ playing the roles of M, f_1, f_2. Therefore.

$$\begin{aligned}
&\mathscr{F} = \{\varphi/\varphi:\omega^2 \times M \dashrightarrow \omega^2 \times M\},\\
&\varphi\psi = \lambda c_1 c_2 r.\psi(\varphi(c_1, c_2, r)),\\
&(\varphi, \psi)(2c_1, c_2, r) = \varphi(c_1, c_2, r),\\
&(\varphi, \psi)(2c_1 + 1, c_2, r) = \psi(c_1, c_2, r),
\end{aligned}$$

while proposition 21.2 implies that

$$\langle \varphi \rangle(\bar{n}(c_1), c_2, r) = (\bar{n}(c_1'), c_2', r') \quad \text{iff} \quad \varphi(c_1, c_2, r) = (c_1', c_2', r'),$$

$\bar{n}$ standing in the sense of example 22.5,
$[\varphi](c_1, c_2, r) = (c_1', c_2', r')$ iff there are n, $(k_0, m_0, r_0), \ldots, (k_n, m_n, r_n)$ such that $(k_0, m_0, r_0) = (c_1, c_2, r)$, k_i is odd and $(k_{i+1}, m_{i+1}, r_{i+1}) = \varphi((k_i - 1)/2, m_i, r_i)$ for all $i < n$, and $(k_n, m_n, r_n) = (2c_1', c_2', r')$. Therefore, the first counter c_1 handles the pairing schemes Π, L, R. The second counter c_2 will be used to implement the operation translation as done in 21.11.

Now the functions we are interested in are members of $\mathscr{F}$. Predicates will also be presented by members of $\mathscr{F}$ assigning $\tilde{P} = (P \to L, R)$ to $P:\omega^2 \times M \dashrightarrow \{0, 1\}$. The following statement characterizes C-computability in the terms of IOS-recursiveness and prime computability of Moschovakis.

Proposition 23.1 (C-Computability Theorem). Let $\mathscr{S}$ be the IOS of example 23.1, $\varphi \in \mathscr{F}$ and B be a unary basis the carrier of which is M. Then the following are equivalent.

(1) φ is C-computable in B.

(2) φ is prime recursive in

$$\mathscr{B} = \{L_2, R_2, K_2, f_1^c, \ldots, f_u^c, \tilde{Z}e, \tilde{E}v_2, \tilde{Z}e_2, \tilde{P}_1^c, \ldots, \tilde{P}_v^c\}.$$

(3) φ is recursive in $\mathscr{B}$.

(4) φ is prime computable in B, i.e., so are its three co-ordinate functions.

Proof. We shall verify (1)$\Leftrightarrow$(2) and (3)$\Rightarrow$(2), while (2)$\Rightarrow$(3) is obvious and (1)$\Leftrightarrow$(4) is established in Soskov [1983].

Let φ be C-computable in B. Then $\varphi = \varphi_1$, where $\varphi_1, \ldots, \varphi_m$ is the least solution to a certain system

(5) $\Gamma_i(\theta_1, \ldots, \theta_m) = \theta_i$, $1 \le i \le m$, such that each mapping Γ_i is of the form

$\lambda\theta.f\theta$ or $\lambda\theta\tau.(P\to\theta,\tau)$ or $\lambda\theta.I$. Noting the equalities

$$(P\to\varphi,\psi)=(P\to L,R)(\varphi,\psi)=\tilde{P}(\varphi,\psi),$$
$$(Ev\to\varphi,\psi)=(L\varphi,R\psi),\quad K=(I,I),$$

the system (5) can be rewritten as $\sigma_i(I,\theta_1,\dots,\theta_m)=\theta_i$, $1\le i\le m$, with certain $\sigma_1,\dots,\sigma_m$ polynomial in $\mathscr{B}$. The latter system is in turn equivalent to the single equality.

(6) $\sigma(I,\theta)=\theta$,

where $\sigma=(\sigma_1,\dots,\sigma_m)$ is polynomial in $\mathscr{B}$. Now, the element $(\varphi_1,\dots,\varphi_m)$ is a least solution to (6). (Cf. the proof of 9.16*.) This implies $(\varphi_1,\dots,\varphi_m)=R[\sigma]$ by 6.11, hence $\varphi=\varphi_1=\bar{1}[\sigma]$ is prime recursive in $\mathscr{B}$. (We even have φ presented in a normal form.)

Conversely, all the members of $\mathscr{F}$ prime recursive in $\mathscr{B}$ are C-computable in B. The program l_1Ll_2, l_2 stop computes L, while similar programs compute $R, L_2, R_2, K_2, f^c_1,\dots,f^c_u$. The program l_1 if Ze then l_2 else l_3, l_2Ll_4, l_3Rl_4, l_4 stop computes $\tilde{Z}e$; similar programs compute $\tilde{E}v_2, \tilde{Z}e_2, \tilde{P}^c_1,\dots,\tilde{P}^c_v$.

Let a program $l_1,\dots,l_m$ compute φ and a program $l_{m+1},\dots,l_{m+k}$ compute ψ, assuming without loss of generality that no label occurs in both programs at the same time.

Replacing throughout the first program the label l_m by l_{m+1}, the program $l_1,\dots,l_{m-1},l_{m+1},\dots,l_{m+k}$ will compute $\varphi\psi$.

Replacing throughout the first program the label l_m by l_{m+k}, the program $l_0,l_1,\dots.l_{m-1},l_{m+1},\dots,l_{m+k+2}$ will compute (φ,ψ), where the new instructions are l_0 if Ev then l_{m+k+1} else l_{m+k+2}, $l_{m+k+1}Kl_1$ and $l_{m+k+2}Kl_{m+1}$.

Replacing throughout the instructions $l_1,\dots,l_{m-1}$ the label l_m by l_0, the function $[\varphi]$ will be computed by the program $l_0,l_1,\dots,l_{m+2}$ the new instructions of which are l_0 if Ev then l_{m+1} else l_{m+2}, $l_{m+1}Kl_m$ and $l_{m+2}Kl_1$. (While the last three assertions seem obvious, this does not mean that they should not be formally proved.)

Suppose now that φ is recursive in $\mathscr{B}$. The elements $W_1=L_2$, $W_2=R_2$ and $W=E\tilde{v}_2K_2$ satisfy the assumptions of 21.10. (Compare with 21.11.) The elements $f^c_1,\dots,f^c_u$, $\tilde{Z}e$, $\tilde{P}^c_1,\dots,\tilde{P}^c_v$ commute with W_1,W_2; the element K_2 commutes with L,R, while the elements $E\tilde{v}_2$, $Z\tilde{e}_2$ satisfy the equalities $L\sigma=\sigma B^2$, $R\sigma=\sigma A^2$. Therefore, φ is prime recursive in $\{W,W_1,W_2\}\cup\mathscr{B}$ by exercise 21.4; hence φ is prime recursive in $\mathscr{B}$. The proof is complete.

It is worth mentioning that the programs constructed when proving the implication (2)$\Rightarrow$(1) above are *structured*, i.e., obtained from simplest programs of the form $\mathscr{P}=l_1fl_2$, l_2 stop by means of consecutive compositions, branchings and loopings. Taken together, the proofs of (1)$\Rightarrow$(2) and (2)$\Rightarrow$(1) ensure that all the unary C-programs can be transformed into equivalent structured ones. (Notice that it suffices to have one counter in order to achieve this.) This seems to capture the essence of the structurization result of Böhm and Jacopini [1966] which will be revisited at the scheme level in chapter 26.

We proceed with the case of n-ary schemes augmented by counters. An *n-ary C-scheme* is a unary C-scheme whose function and predicate symbols are among the following:

$$I_i^{(j)}, \quad 1 \le i \ne j \le n,$$
$$f_{1,i_1,\dots,i_n}^{(j)}, f_{2,i_1,\dots,i_n}^{(j)}, \dots, 1 \le i_1, \dots, i_n, j \le n,$$
$$P_{1,i_1,\dots,i_n}, P_{2,i_1,\dots,i_n}, \dots, 1 \le i_1, \dots, i_n \le n,$$

plus constants operating the counters.

A *n-ary basis* is a tuple $B_n = (M, f_1, \dots, f_u, P_1, \dots, P_v)$, where

$$f_i : M^{m_i} \longrightarrow M, 1 \le m_i \le n,$$
$$P_i : M^{k_i} \longrightarrow \{0, 1\}, 1 \le k_i \le n.$$

A n-ary C-scheme is now interpreted by a unary modification of B_n,

$$B_n' = (M^n, I_i^{(j)}, 1 \le i \ne j \le n, f_{1,i_1,\dots,i_n}^{(j)}, \dots, f_{u,i_1,\dots,i_n}^{(j)},$$
$$P_{1,i_1,\dots,i_n}, \dots, P_{v,i_1,\dots,i_n}, 1 \le i_1, \dots, i_n, j \le n),$$

where

$$I_i^{(j)} = \lambda r_1 \dots r_n . (r_1, \dots, r_{j-1}, r_i, r_{j+1}, \dots, r_n),$$
$$f_{i,i_1,\dots,i_n}^{(j)} = \lambda r_1 \dots r_n . (r_1, \dots, r_{j-1}, f_i(r_{i_1}, \dots, r_{i_{m_i}}), r_{j+1}, \dots, r_n),$$
$$P_{i,i_1,\dots,i_n} = \lambda r_1 \dots r_n . P_i(r_{i_1}, \dots, r_{i_{k_i}}).$$

Therefore, the *n-ary* C-*programs* in B_n are just unary C-programs in B_n' and their execution requires two counters and n registers. Consequently, a function $\varphi : \omega^2 \times M^n \longrightarrow \omega^2 \times M^n$ is C-*computable in* B_n iff φ is C-computable in the unary basis B_n'. This notion of C-computability is characterized in terms of IOS as follows.

Example 23.2. Example 23.1 with M replaced by M^n.

Proposition 23.2. Let $\mathscr{S}$ be the IOS of example 23.2, $\varphi \in \mathscr{F}$ and let B_n be a n-ary basis with carrier M. Then the following are equivalent.

(1) φ is C-computable in B_n.

(2) φ is prime recursive in

$$\mathscr{B}_n = \{L_2, R_2, K_2, I_i^{(j)c}, 1 \le i \ne j \le n, f_{1,i_1,\dots,i_n}^{(j)c}, \dots, f_{u,i_1,\dots,i_n}^{(j)c},$$
$$\widetilde{Ze}, \widetilde{Ev}_2, \widetilde{Ze}_2, \widetilde{P^c}_{1,i_1,\dots,i_n}, \dots, \widetilde{P^c}_{v,i_1,\dots,i_n}, 1 \le i_1, \dots, i_n, j \le n\}.$$

(3) φ is recursive in $\mathscr{B}_n$.

This follows from 23.1.

If all the functions of B_n are unary, then C-computability in B_n is again equivalent to prime computability in B_n, as shown in Soskov [1983]. Otherwise this equivalence may fail since one needs a *stack* in order to compute arbitrary functions prime computable in B_n. Thus we come to the next kind of scheme to be considered, viz., schemes with counting and stacking, which we shall call CS-schemes.

The *n-ary* CS-*schemes* are unary C-schemes whose function symbols include the constants $Si_1, \dots, Si_n, So_1, \dots, So_n$ (Si for 'stack-in', So for 'stack-out'). An extra predicate constant may also be added to check on the emptiness of the stack. Given a n-ary basis $B_n = (M, f_1, \ldots, f_u, P_1, \ldots, P_v)$, such a scheme is interpreted by means of a unary modification of B_n,

$$B_n^s = (M^n \times M^*, Si_1, \ldots, Si_n, So_1, \ldots, So_n, f_1^s, \ldots, f_u^s, P_1^s, \ldots, P_v^s),$$

where $M^* = \bigcup_n M^n$ and, x ranging over M^*,

$$\begin{aligned}
&Si_i = \lambda r_1 \ldots r_n x.(r_1, \ldots, r_n, r_i, x),\\
&So_i(r_1, \ldots, r_n) = (r_1, \ldots, r_n),\\
&So_i(r_1, \ldots, r_n, r, x) = (r_1, \ldots, r_{i-1}, r, r_{i+1}, \ldots, r_n, x)\\
&f_i^s = \lambda r_1 \ldots r_n x.(f_i(r_1, \ldots, r_{m_i}), r_2, \ldots, r_n, x),\\
&P_i^s = \lambda r_1 \ldots r_n x.P_i(r_1, \ldots, r_{k_i}).
\end{aligned}$$

Correspondingly, the *n-ary* CS-*programs* in B_n are exactly the unary C-programs in B_n^s and a function $\varphi: \omega^2 \times M^n \times M^* \longrightarrow \omega^2 \times M^n \times M^*$ is CS-*computable in* B_n iff it is C-computable in B_n^s.

Example 23.3. Example 23.1 with M replaced by $M^n \times M^*$.

Proposition 23.3. Let $\mathscr{S}$ be the IOS of example 23.3, $\varphi \in \mathscr{F}$ and let B_n be a n-ary basis with carrier M. Then the following are equivalent.

(1) φ is CS-computable in B_n.

(2) φ is prime recursive in

$$\mathscr{B}_n = \{L_2, R_2, K_2, Si_i^c, So_i^c, 1 \leq i \leq n, f_1^{sc}, \ldots, f_u^{sc}, \widetilde{Ze}, \widetilde{Ev}_2, \widetilde{Ze}_2, \widetilde{P_1^{sc}}, \ldots, \widetilde{P_v^{sc}}\}.$$

(3) φ is recursive in $\mathscr{B}_n$.

This follows from 23.1.

Usually the content of the stack is regarded neither as an input not as an output. That is, one retrieves from the stack only what has been loaded there earlier and, on the other hand, programs can be composed in such way that data loaded in the stack is later retrieved.

Example 23.2 is isomorphic to the subspace of example 23.3, assigning $\varphi^s = \lambda c_1 c_2 r_1 \ldots r_n x.(\varphi(c_1, c_2, r_1, \ldots, r_n), x)$ to $\varphi: \omega^2 \times M^n \longrightarrow \omega^2 \times M^n$. The functions $I_i^{(j)cs}$ may be computed by making use of the stack since $I_i^{(j)cs} = Si_i^c So_j^c$. It also follows that $f_{i,i_1,\ldots,i_n}^{(j)cs}$, $(\widetilde{P_{i,i_1,\ldots,i_n}^c})^s$ can be obtained by multiplying f_i^{sc}, $\widetilde{P_i^{sc}}$, $Si_1, \ldots, Si_n$, $So_1, \ldots, So_n$; hence whenever φ is C-computable in B_n, φ^s is CS-computable in B_n. According to Moldestad, Stoltenberg-Hansen and Tucker [1981] the last implication can be reversed, provided all the functions of B_n are unary. The same holds whenever M is finite (I. Soskov).

Functions over $\omega^2 \times M^n$ CS-computable in B_n are characterized as follows.

Proposition 23.4. Let B_n be a n-ary basis, let $\mathscr{B}_n$ correspond to it as in the previous statement and let $\varphi: \omega^2 \times M^n \longrightarrow \omega^2 \times M^n$. Then the following are equivalent.

(1) φ is CS-computable in B_n, i.e., so is φ^s.

(2) φ^s is prime recursive in $\mathscr{B}_n$.

(3) φ^s is recursive in $\mathscr{B}_n$.

(4) φ is prime computable in B_n, i.e., so are its co-ordinate functions.

Proof. The assertions (1), (2), (3) are equivalent by 23.3, while (1)$\Leftrightarrow$(4) is proved in Soskov [1983].

Certain kinds of recursive schemes and programs may also be considered. These are systems of equalities

$$(1) \qquad \Gamma_i(\theta_1,\dots,\theta_m)=\theta_i,\ 1\le i\le m,$$

with arbitrary mappings Γ_i constructed by the operations of multiplication and branching. In this way one gets *unary recursive* C-*schemes*, *n-ary recursive* C-*schemes* and *n-ary recursive* CS-*schemes*, depending on the function and predicate constants allowed. If the function and predicate symbols are interpreted, then we have *recursive programs*. The function $\varphi_{\mathscr{P}}$ computed by such a program $\mathscr{P}$ is by definition the first component of the least solution to the corresponding system of equalities. However, recursion can be eliminated since the First Recursion Theorem enables us to solve equations.

Proposition 23.5 (Recursion Elimination Theorem). Let $\mathscr{P}$ be a unary recursive C-program (respectively n-ary recursive C-program, n-ary recursive CS-program). Then there is a unary C-program (n-ary C-program, n-ary CS-program) $\mathscr{P}_1$ in the same basis such that $\varphi_{\mathscr{P}}=\varphi_{\mathscr{P}_1}$.

Proof. Following the proof of 23.1, we rewrite the system (1) as a system introduced by mappings polynomial in the corresponding set $\mathscr{B}$ (respectively $\mathscr{B}_n$). The latter system is then solved by making use of 9.16* and finally the implication (3)$\Rightarrow$(1) of 23.1 (respectively 23.2, 23.3) completes the proof.

In fact, the above recursion elimination (i.e., implementation) algorithm does not depend on interpretations, so the result holds for schemes as well. The case of unary C-schemes is considered in Greibach [1975], while it is well known in Computer Science that recursion can be implemented by stacking. It should be stressed that the availability of $I_i^{(j)}$ or Si_i, So_i determines the particular n-ary recursive schemes which one can actually write down. (Cf. exercise 23.4.)

In another development, one may consider *multiple-valued program schemes*. The definitions are the same but function and predicate symbols are interpreted by multiple-valued functions and predicates. Analogues of 23.1–23.5 hold with $\mathscr{S}$ obtained from example 22.1 rather than 22.2.

A natural multiple-valued function constant to be allowed is $U=\lambda c_1c_2r.\{(2c_1,c_2,r),(2c_1+1,c_2,r)\}$ or, equivalently, $\lambda c_1c_2r.\{(0,c_2,r),(1,c_2,r)\}$ or $\lambda c_1c_2r.\{(n,c_2,r)/n\in\omega\}$, or the predicate constant $\lambda c_1c_2r.\{0,1\}$. The same effect would be achieved by lifting the restriction that each label in a scheme appear only once as a prefix. If the scheme consists of m instructions in which k distinct labels occur, $k\le m$, then the corresponding characteristic system is of the form

$$\Gamma_j(\theta_1,\dots,\theta_k)\le\theta_{i_j},\quad 1\le j\le m,$$

where $1\le i_1,\dots,i_m\le k$. According to exercise 9.4 such systems may be solved by making use of U while, conversely, U may be computed by the program l_1Ll_2, l_1Rl_2, l_2 stop. In the case of unary C-schemes and n-ary CS-schemes the computability so obtained is equivalent to prime computability in B, U, respectively B_n, U. As pointed out in Soskov [1983], the latter is in essence computability in B or B_n by the effective definitional schemes of Friedman [1971], a natural analogue to relative partial recursiveness.

Finally, *infinitary schemes* may also be considered. These are infinite lists of instructions, one of them being l_m stop. Constants L_2, R_2, K_2, Ev_2 and Ze_2 are not needed since one counter proves sufficient now. The corresponding notions of infinitary C-computability and infinitary CS-computability can be characterized by means of (prime) c-recursiveness in appropriate IOS.

EXERCISES TO CHAPTER 23

Exercise 23.1. Let $\mathscr{P} = l_1, \ldots, l_m$ be a unary program, let $\varphi_{\mathscr{P}}$ be the function computed by $\mathscr{P}$ and let $\tau_1, \ldots, \tau_m$ be a solution to the characteristic system of $\mathscr{P}$. Show that $\varphi_{\mathscr{P}} \leq \tau_1$.

Hint. Let $\varphi_{\mathscr{P}}(s) = t$. Then there are $n, i_0, \ldots, i_n$, $r_0, \ldots, r_n$ such that $i_0 = 1$, $i_n = m$, $r_0 = s$, $r_n = t$ and for all $k < n$ either $l_{i_k} f l_{i_{k+1}} \in \mathscr{P}$ and $r_{k+1} = f(r_k)$ or if P then l_i else $l_j \in \mathscr{P}$, $r_{k+1} = r_k$ and either $P(r_k) = 0$, $i_{k+1} = i$ or $P(r_k) = 1$, $i_{k+1} = j$. Starting with $k = n$ show that $\tau_{i_k}(r_k) = t$ for all $k \leq n$. In particular, $\tau_1(s) = t$.

Example 23.4. Example 22.2 with $\omega \times M$ taken for M, $L = \lambda c_1 r.(2c_1, r)$ and $R = \lambda c_1 r.(2c_1 + 1, r)$.

Exercise 23.2. Let $\mathscr{S}$ be the IOS of example 23.4, let $B = (M, f_1, \ldots, f_u, P_1, \ldots, P_v)$ be a unary basis, let $\varphi \in \mathscr{F}$ and let $\varphi^*: \omega^2 \times M \dashrightarrow \omega^2 \times M$ be such that $\varphi^*(c_1, c_2, r) = (c_1', c_2', r')$ iff $c_2' = c_2$ and $\varphi(c_1, r) = (c_1', r')$. Show that φ^* is C-computable in B iff φ is recursive in $\mathscr{B} = \{\lambda c_1 r.(c_1, f_i(r)), 1 \leq i \leq u, (\lambda c_1 r.\mathrm{sg} c_1)\breve{}, (\lambda c_1 r.P_i(r))\breve{}, 1 \leq i \leq v\}$.

Hint. Let $\mathscr{S}_1$ be the IOS of example 23.1. Then φ^* is C-computable in B iff it is recursive in a corresponding subset of $\mathscr{F}_1$ by 23.1. Using exercise 21.6, show that the latter is equivalent to φ being recursive in $\mathscr{B}$.

Exercise 23.3. Let $\mathscr{S}$ be the IOS of example 22.2, i.e. example 4.7. Show that φ is recursive in $\psi_1, \ldots, \psi_m$ iff $\lambda c_1 c_2 r.(c_1, c_2, \varphi(r))$ is C-computable in $B = (M, f_1, f_2, f_1^{-1}, f_2^{-1}, \psi_1, \ldots, \psi_m, Ev)$, where Ev is a predicate true of $f_1(M)$, false on $f_2(M)$ and undefined otherwise.

Hint. Let $\mathscr{S}_1$ be the IOS of example 22.2 with $\omega \times M$ taken for M, preserving the splitting scheme of M, and let $\mathscr{S}_2$ be the IOS of example 23.4. Apply exercise 21.6 to $\mathscr{S}, \mathscr{S}_1$, exercise 7.2 to $\mathscr{S}_1, \mathscr{S}_2$ and exercise 23.2 to $\mathscr{S}_2$.

Remark. By 23.1 φ is recursive in $\psi_1, \ldots, \psi_m$ iff φ is prime computable in $f_1, f_2, f_1^{-1}, f_2^{-1}$, $\psi_1, \ldots, \psi_m, Ev$. The relative recursiveness of example 4.8 is characterized in the same way since exercise 23.2, 23.3 can be restated for multiple-valued functions.

Exercise 23.4. Let $g, g_1, g_2: M \dashrightarrow M$, $h: M^2 \dashrightarrow M$ and $P: M \dashrightarrow \{0, 1\}$. Write recursive programs to compute the functions f introduced by recursion as follows.

$$\text{a.}\quad f(r)=\begin{cases} g(r), & \text{if } P(r)=0,\\ g_2(f(g_1(r))), & \text{if } P(r)=1,\\ \uparrow, & \text{otherwise.}\end{cases}$$

$$\text{b.}\quad f(r)=\begin{cases} g(r), & \text{if } P(r)=0,\\ h(f(g_1(r)),r), & \text{if } P(r)=1,\\ \uparrow, & \text{otherwise.}\end{cases}$$

$$\text{c.}\quad f(r)=\begin{cases} g(r), & \text{if } P(r)=0,\\ h(f(g_1(r)),f(g_2(r))), & \text{if } P(r)=1,\\ \uparrow, & \text{otherwise.}\end{cases}$$

Hint. a. $(P\to g, g_1\theta g_2)=\theta$.

b. $(P^s\to g^s, Si_2Si_1So_2g_1^s\theta h^sSo_2)=\theta$.

c. $(P^s\to g^s, Si_2Si_1So_2g_2^s\theta Si_1Si_2So_1So_2g_1^s\theta h^sSo_2)=\theta$.

Remark. Superscripts c turn the last two binary recursive programs with stacking into binary recursive CS-programs which can be transformed into equivalent binary CS-programs by 23.5. More generally, this exercise shows how arbitrary polyadic recursive procedures can be transformed into equivalent recursive CS-schemes (cf. Skordev [1983]).

CHAPTER 24

Prime and search computability

The best known concepts of effectively computable functions over arbitrary domains are perhaps those of prime and search computability of Moschovakis. This chapter is aimed to show that they are particular instances of t-recursiveness.

We begin with some definitions from Moschovakis [1969] adduced in slightly modified notations.

Let M be an arbitrary set the members of which are not ordered pairs, $0 \notin M$ and

$$N^* = cl(N \cup \{0\}/\lambda st.(s,t))$$

whenever $N \subseteq M$. Natural numbers are introduced by taking $n+1=(0,n)$, while $\langle s_1,\ldots,s_n\rangle$ stands for $(n,(s_1,(s_2,\ldots,(s_n,0)\ldots)))$ and $L^*, R^*: M^* \to M^*$ such that $L^*(0)=R^*(0)=0$, $L^*(M)=R^*(M)=1$ and $L^*((s,t))=s$, $R^*((s,t))=t$. Object of interest are the n-ary partial multiple-valued functions over M^*, $\varphi: M^{*n} \to 2^{M^*}$.

The basic notion of prime computability is introduced via index schemes following the pattern of Kleene [1959]. Given a subset N of M^* and a list of functions $\psi_1,\ldots,\psi_l$, ψ_i being n_i-ary, an n-ary function φ is *prime computable with constants derived from N in* $\psi_1,\ldots,\psi_l$, denoted $\varphi \in PC(N,\psi_1,\ldots,\psi_l)$, iff $\varphi = \lambda s_1 \ldots s_n.\sigma(e,s_1,\ldots,s_n)$ for a certain index $e \in N^*$. The universal function σ is the least fixed point of the mapping $\Omega: \mathscr{F}_0 \to \mathscr{F}_0$, where $\mathscr{F}_0 = \{\varphi_0/\varphi_0: \bigcup_{n>0} M^{*n} \to 2^{M^*}\}$ and $\Omega(\varphi_0)$ is defined by the following clauses for all $\varphi_0 \in \mathscr{F}_0$.

0. If $t \in \psi_i(s_1,\ldots,s_{n_i})$, $1 \le i \le l$, then $t \in \Omega(\varphi_0)(\langle 0, n_i+n, i\rangle, s_1,\ldots,s_{n_i}, r_1,\ldots,r_n)$.

1. $t \in \Omega(\varphi_0)(\langle 1,n,t\rangle, s_1,\ldots,s_n)$.

2. $t \in \Omega(\varphi_0)(\langle 2,n+1\rangle, t, s_1,\ldots,s_n)$.

3. $(t,r) \in \Omega(\varphi_0)(\langle 3,n+2\rangle, t, r, s_1,\ldots,s_n)$.

4_0. $L^*(t) \in \Omega(\varphi_0)(\langle 4,n+1,0\rangle, t, s_1,\ldots,s_n)$.

4_1. $R^*(t) \in \Omega(\varphi_0)(\langle 4,n+1,1\rangle, t, s_1,\ldots,s_n)$.

5. If $t \in \varphi_0(e_1,r,s_1,\ldots,s_n)$ for a certain $r \in \varphi_0(e_2,s_1,\ldots,s_n)$, then $t \in \Omega(\varphi_0)(\langle 5,n,e_1,e_2\rangle, s_1,\ldots,s_n)$.

6. If $r \in M \cup \{0\}$ and $t \in \varphi_0(e_1,r,s_1,\ldots,s_n)$, then $t \in \Omega(\varphi_0)(\langle 6,n+1,e_1,e_2\rangle, r, s_1,\ldots,s_n)$.

If $t \in \varphi_0(e_2, u, v, s, r, s_1, \ldots, s_n)$ for certain $u \in \varphi_0(\langle 6, n+1, e_1, e_2 \rangle, s, s_1, \ldots, s_n)$ and $v \in \varphi_0(\langle 6, n+1, e_1, e_2 \rangle, r, s_1, \ldots, s_n)$, then $t \in \Omega(\varphi_0)(\langle 6, n+1, e_1, e_2 \rangle, (s, r), s_1, \ldots, s_n)$.

7. If $t \in \varphi_0(e, s_{j+1}, s_1, \ldots, s_j, s_{j+2}, \ldots, s_n)$, then $t \in \Omega(\varphi_0)(\langle 7, n, j, e \rangle, s_1, \ldots, s_n)$.

8. If $t \in \varphi_0(e, s_1, \ldots, s_n)$, then $t \in \Omega(\varphi_0)(\langle 8, n+m+1, n \rangle, e, s_1, \ldots, s_n, r_1, \ldots, r_m)$.

Notice that Ω is a continuous mapping, hence $\sigma = \sup_n \sigma_n$, where σ_0 is the nowhere defined function and $\sigma_{n+1} = \Omega(\sigma_n)$. Traditionally, $\{e\}(s_1, \ldots, s_n) \to t$ is written for $t \in \sigma(e, s_1, \ldots, s_n)$. If $\varphi \in PC(\varnothing, \psi_1, \ldots, \psi_l)$, then φ is *prime computable in* $\psi_1, \ldots, \psi_l$. Some of the functions $\psi_1, \ldots, \psi_l$ may be regarded as predicates, provided they are $\{0,1\}$-valued.

Let us design an appropriate IOS to express the notion of prime computability. It suffices to consider a space composed of unary functions over M^* since this set is closed under the prime computable pairing function $\lambda st.(s,t)$.

Example 24.1. The IOS $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ of example 22.1 (alias example 4.8) with M^* taken for $M, L = \lambda s.(0,s)$ and $R = \lambda s.(1,s)$. Then, $\mathscr{F} = \{\varphi/\varphi\colon M^* \to 2^{M^*}\}$, $I = \lambda s.s$, $\varphi \le \psi$ iff $\varphi \subseteq \psi$, $\varphi\psi = \lambda s.\psi(\varphi(s))$, $(\varphi, \psi)((0,s)) = \varphi(s)$, $(\varphi, \psi)((1,s)) = \psi(s)$ and $(\varphi, \psi)(s)\uparrow$ otherwise.

Proposition 21.2 characterizes the operations $\langle\ \ \rangle$, $[\ \ ]$ explicitly as follows.

$t \in \langle \varphi \rangle(s)$ iff there are n, s_0, t_0 such that $s = \bar{n}(s_0)$, $t = \bar{n}(t_0)$ and $t_0 \in \varphi(s_0)$.

$t \in [\varphi](s)$ iff there are $n, r_0, \ldots, r_n, t_0, \ldots, t_{n-1}$ such that $r_0 = s$, $r_i = (1, t_i)$ and $r_{i+1} \in \varphi(t_i)$ for all $i < n$, while $r_n = (0, t)$.

A storing operation corresponds to the pairing function $J = \lambda st.(s,t)$ by 21.13. Namely,

$$St(\varphi)((s,t)) = (s, \varphi(t))$$

and $St(\varphi)(s)\uparrow$ otherwise. We broaden the notion of st-recursiveness by assuming the following elements $K_0^* - K_3^*$ initial:

$K_0^* = \lambda s.(s,s)$, $K_1^*((s,t)) = (t,s)$ and $K_1^*(s)\uparrow$ otherwise, $K_2^*(0)$, $K_3^*(M) = 0$ and $K_2^*(s)$, $K_3^*(s) = 1$ otherwise. Then the elements $K_0 - K_6$ of the proof of 21.13 and exercise 21.7 are expressed as follows. One may take $K_3 = L$, while

$$\begin{aligned}
&K_4 = K_1^* St(K_2^*) K_1^*(I, I), \qquad K_5 = K_0^*,\\
&K_6 = K_0^* St(K_4 K_1^* K_4) K_1^* St^2(K_4),\\
&K_0 = K_6(L, R), \qquad K_1 = St(K_1^*) K_6 K_1^*,\\
&K_2 = K_1^* K_6 St(K_1^*).
\end{aligned}$$

Therefore, an element φ is now st-recursive in $\mathscr{B}$ iff $\varphi \in cl(\{L, R, K_0^* - K_3^*\} \cup \mathscr{B}/\circ, \Pi, \langle\ \ \rangle, [\ \ ], St)$.

We recall that the set $\mathscr{L}$ of St is $M^{\sim} = \{\tilde{s} = \lambda t.(s,t)/s \in M\}$ and assign to each $\varphi_0 \in \mathscr{F}_0$ a unary function $\varphi_0^{\sim} \in \mathscr{F}$ such that $t \in \varphi_0^{\sim}(s)$ iff $\exists n s_1 \ldots s_n (s = \langle s_1, \ldots, s_n \rangle \,\&\, t \in \varphi_0(s_1, \ldots, s_n))$.

Proposition 24.1. If $\phi \in PC(N, \psi_1, \ldots, \psi_l)$, then φ is st-recursive in $N^{\sim} \cup \{\tilde{\psi}_1, \ldots, \tilde{\psi}_l\}$.

Proof. There is an $e \in N^*$ such that $\varphi = \lambda s_1 \ldots s_n . \sigma(e, s_1, \ldots, s_n)$, hence $\tilde{\varphi} = \tilde{e} K_6 K_1^* St(L) K_1^* \tilde{\sigma}$. The element $\tilde{e}$ is st-recursive in $N^{\sim}$, hence it suffices to prove that $\tilde{\sigma}$ is st-recursive in $\tilde{\psi}_1, \ldots, \tilde{\psi}_l$. For this purpose we construct a mapping $\Omega^{\sim} : \mathscr{F} \to \mathscr{F}$ st-recursive in $\tilde{\psi}_1, \ldots, \tilde{\psi}_l$ such that $\tilde{\sigma} = \mu\theta . \Omega^{\sim}(\theta)$. Namely, take

$$\Omega^{\sim} = \lambda\theta . \rho((\tilde{\psi}_1, \ldots, \tilde{\psi}_l, I), I, St(\theta)\rho_1\theta, \theta, St(St(\theta)K_1^* St(\theta))\rho_2\theta),$$

where

$$\rho(\langle\langle 0, n_i + n, i\rangle, s_1, \ldots, s_{n_i}, r_1, \ldots, r_n\rangle) = \bar{i}\bar{0}(\langle s_1, \ldots, s_{n_i}\rangle), \quad 1 \le i \le l,$$
$$\rho(\langle\langle 1, n, t\rangle, s_1, \ldots, s_n\rangle) = \bar{1}(t),$$
$$\rho(\langle\langle 2, n+1\rangle, t, s_1, \ldots, s_n\rangle) = \bar{1}(t),$$
$$\rho(\langle\langle 3, n+2\rangle, t, r, s_1, \ldots, s_n\rangle) = \bar{1}((t, r)),$$
$$\rho(\langle\langle 4, n+1, 0\rangle, t, s_1, \ldots, s_n\rangle) = \bar{1}(L^*(t)),$$
$$\rho(\langle\langle 4, n+1, 1\rangle, t, s_1, \ldots, s_n\rangle) = \bar{1}(R^*(t)),$$
$$\rho(\langle\langle 5, n, e_1, e_2\rangle, s_1, \ldots, s_n\rangle) = \bar{2}((\langle e_1, s_1, \ldots, s_n\rangle, \langle e_2, s_1, \ldots, s_n\rangle)),$$
$$\rho(\langle\langle 6, n+1, e_1, e_2\rangle, r, s_1, \ldots, s_n\rangle) = \bar{3}(\langle e_1, r, s_1, \ldots, s_n\rangle), \quad \text{if } r \in M \cup \{0\},$$
$$\rho(\langle\langle 6, n+1, e_1, e_2\rangle, (s, r), s_1, \ldots, s_n\rangle) = R^4((\langle e_2, s, r, s_1, \ldots, s_n\rangle,$$
$$(\langle\langle 6, n+1, e_1, e_2\rangle, s, s_1, \ldots, s_n\rangle, \langle\langle 6, n+1, e_1, e_2\rangle, r, s_1, \ldots, s_n\rangle))),$$
$$\rho(\langle\langle 7, n, j, e\rangle, s_1, \ldots, s_n\rangle) = \bar{3}(\langle e, s_{j+1}, s_1, \ldots, s_j, s_{j+2}, \ldots, s_n\rangle),$$
$$\rho(\langle\langle 8, n+m+1, n\rangle, e, s_1, \ldots, s_n, r_1, \ldots, r_m\rangle) = \bar{3}(\langle e, s_1, \ldots, s_n\rangle)$$

and $\rho(s)\uparrow$ otherwise,

$$\rho_1((\langle e, s_1, \ldots, s_n\rangle, r)) = \langle e, r, s_1, \ldots, s_n\rangle,$$
$$\rho_2((\langle e, s, r, s_1, \ldots, s_n\rangle, (v, u))) = \langle e, u, v, s, r, s_1, \ldots, s_n\rangle$$

and $\rho_1(s)$, $\rho_2(s)\uparrow$ otherwise. Therefore, for all θ the element $\Omega^{\sim}(\theta)$ is defined as follows.

0. If $t \in \tilde{\psi}_i(\langle s_1, \ldots, s_{n_i}\rangle)$, $1 \le i \le l$, then $t \in \Omega^{\sim}(\theta)(\langle\langle 0, n_i + n, i\rangle, s_1, \ldots, s_{n_i}, r_1, \ldots, r_n\rangle)$.

1. $t \in \Omega^{\sim}(\theta)(\langle\langle 1, n, t\rangle, s_1, \ldots, s_n\rangle)$.

2. $t \in \Omega^{\sim}(\theta)(\langle\langle 2, n+1\rangle, t, s_1, \ldots, s_n\rangle)$.

3. $(t, r) \in \Omega^{\sim}(\theta)(\langle\langle 3, n+2\rangle, t, r, s_1, \ldots, s_n\rangle)$.

4_0. $L^*(t) \in \Omega^{\sim}(\theta)(\langle\langle 4, n+1, 0\rangle, t, s_1, \ldots, s_n\rangle)$.

4_1. $R^*(t) \in \Omega^{\sim}(\theta)(\langle\langle 4, n+1, 1\rangle, t, s_1, \ldots, s_n\rangle)$.

5. If $t \in \theta(\langle e_1, r, s_1, \ldots, s_n\rangle)$ for a certain $r \in \theta(\langle e_2, s_1, \ldots, s_n\rangle)$, then

$$t \in \Omega^{\sim}(\theta)(\langle\langle 5, n, e_1, e_2\rangle, s_1, \ldots, s_n\rangle).$$

6. If $t \in \theta(\langle e_1, r, s_1, \ldots, s_n\rangle)$, $r \in M \cup \{0\}$, then $t \in \Omega^{\sim}(\theta)(\langle\langle 6, n+1, e_1, e_2\rangle, r, s_1, \ldots, s_n\rangle)$.

If $t \in \theta(\langle e_2, u, v, s, r, s_1, \ldots, s_n\rangle)$ for certain $u \in \theta(\langle\langle 6, n+1, e_1, e_2\rangle, s, s_1, \ldots, s_n\rangle)$, $v \in \theta(\langle\langle 6, n+1, e_1, e_2\rangle, r, s_1, \ldots, s_n\rangle)$, then

$$t \in \Omega^{\sim}(\theta)(\langle\langle 6, n+1, e_1, e_2\rangle, (s, r), s_1, \ldots, s_n\rangle).$$

7. If $t \in \theta(\langle e, s_{j+1}, s_1, \ldots, s_j, s_{j+2}, \ldots, s_n\rangle)$, then

$$t \in \Omega^{\sim}(\theta)(\langle\langle 7, n, j, e\rangle, s_1, \ldots, s_n\rangle).$$

8. If $t \in \theta(\langle e, s_1, \dots, s_n \rangle)$, then

$$t \in \Omega^{\sim}(\theta)(\langle\langle 8, n+m+1, n\rangle, e, s_1, \dots, s_n, r_1, \dots, r_m\rangle).$$

It follows in particular that $\Omega^{\sim}(\tilde{\varphi}_0) = \Omega(\varphi_0)^{\sim}$ for all $\alpha_0 \in \mathscr{F}_0$; hence

$$\tilde{\sigma} = (\sup_n \Omega^n(O_0))^{\sim} = \sup_n \Omega^n(O_0)^{\sim} = \sup_n \Omega^{\sim n}(\tilde{O_0})$$

$$= \sup_n \Omega^{\sim n}(O) = \mu\theta . \Omega^{\sim}(\theta).$$

The elements ρ, ρ_1, ρ_2 are st-recursive (further details about this are left to the exercises), hence $\Omega^{\sim}$ is st-recursive in $\tilde{\psi}_1, \dots, \tilde{\psi}_l$. Therefore, $\tilde{\sigma}$ is st-recursive in $\tilde{\psi}_1, \dots, \tilde{\psi}_l$ by 10.8*, which completes the proof.

Proposition 24.2. If φ is a n-ary function and $\tilde{\varphi}$ is st-recursive in $N^{\sim} \cup \{\tilde{\psi}_1, \dots, \tilde{\psi}_l\}$, then $\varphi \in PC(N, \psi_1, \dots, \psi_l)$.

Proof. The function φ is prime computable in $\tilde{\varphi}$; hence it suffices to show that whenever $\varphi \in \mathscr{F}$ is st-recursive in $N^{\sim} \cup \{\tilde{\psi}_1, \dots, \tilde{\psi}_l\}$, then $\varphi \in PC(N, \psi_1, \dots, \psi_l)$.

The initial functions $L, R, K_0^* - K_3^*, \tilde{\psi}_1, \dots, \tilde{\psi}_l$ and $\tilde{e}$, $e \in N$, are in $PC(N, \psi_1, \dots, \psi_l)$, and it is immediate that the set $PC(N, \psi_1, \dots, \psi_l)$ is closed under the operations $\circ, \Pi, St$. It remains to show that this set is closed under $\langle\ \ \rangle$, $[\ \ ]$ as well. The case of iteration follows.

Suppose that $\varphi \in PC(N, \psi_1, \dots, \psi_l)$, i.e. $\varphi = \lambda s . \sigma(e_0, s)$ with a certain $e_0 \in N^*$. Then there is an index $e_1 \in N^*$ such that $\lambda s . \sigma(e_1, e, s) = (I, \varphi \lambda s . \sigma(e, s))$ for all e and $\lambda s . \sigma_{n+1}(e_1, e, s) \leq (I, \varphi \lambda s . \sigma_n(e, s))$ for all e, n. (We recall that $\sigma_n = \Omega^n(O_0)$.)

There is by lemma 21 of Moschovakis [1969] an index $e \in N^*$ such that $\sigma(e, s) = \sigma(e_1, e, s)$ for all s and $\sigma_n(e, s) \subseteq \sigma_n(e_1, e, s)$ for all s, n. Writing ψ for $\lambda s . \sigma(e, s)$, it follows that $\psi = (I, \varphi\psi)$, hence $[\varphi] \leq \psi$. On the other hand, $\lambda s . \sigma_0(e, s) \leq [\varphi]$ and whenever $\lambda s . \sigma_n(e, s) \leq [\varphi]$, then

$$\lambda s . \sigma_{n+1}(e, s) \leq \lambda s . \sigma_{n+1}(e_1, e, s) \leq (I, \varphi \lambda s . \sigma_n(e, s)) \leq (I, \varphi[\varphi]) = [\varphi],$$

hence

$$\psi = \lambda s . \sigma(e, s) = \sup_n \lambda s . \sigma_n(e, s) \leq [\varphi].$$

Therefore, $[\varphi] = \psi \in PC(N, \psi_1, \dots, \psi_l)$.

The case of translation is treated similarly, though it may be omitted because the operation $\langle\ \ \rangle$ is prime st-recursive. (This will follow from the proof of 27.17; see also exercise 21.7.) This completes the proof.

The following Prime Computability Theorem characterizes the prime computability of Moschovakis in terms of st-recursiveness. A similar characterization may be found in Skordev [1980].

Proposition 24.3. Let φ be a n-ary function over M^*. Then the following are equivalent.

(1) $\varphi \in PC(N, \psi_1, \dots, \psi_l)$.
(2) $\tilde{\varphi}$ is st-recursive in $N^{\sim} \cup \{\tilde{\psi}_1, \dots, \tilde{\psi}_l\}$.
(3) $\tilde{\varphi}$ is prime st-recursive in $N^{\sim} \cup \{\tilde{\psi}_1, \dots, \tilde{\psi}_l\}$.

In particular, if $\varphi, \psi_1, \ldots, \psi_l$ are unary, then $\varphi, \psi_1, \ldots, \psi_l$ can be substituted for $\tilde{\varphi}, \tilde{\psi}_1, \ldots, \tilde{\psi}_l$ in (2), (3).

This follows from 24.1, 24.2 and the fact that, as mentioned in the proof of 24.2, translation is prime st-recursive.

One may substitute $\check{N} = \{\check{s} = \lambda t.s / s \in N\}$ for $N^{\sim}$ since $\tilde{s} = K_0^* St(\check{s}) K_1^*$ and $\check{s} = \tilde{s} K_1^* K_4$. The set $N^{\sim}$ may also be replaced by $\mathscr{L}_0 = N^{*\sim}$ and obviously $\mathscr{L}_0^2 K_1 \subseteq \mathscr{L}_0$, hence the 'boldface' version results mentioned in the remark to exercise 10.9 hold for st-recursiveness in $N^{*\sim} \cup \{\tilde{\psi}_1, \ldots, \tilde{\psi}_l\}$.

If $\psi_1, \ldots, \psi_l$ are single-valued, then the above constructions can be carried out within the subspace of example 24.1 consisting of all the unary single-valued functions over M^*, which is a particular instance of example 22.2.

The characterization of the search computability of Moschovakis is now immediate since

$$SC(N, \psi_1, \ldots, \psi_l) = PC(N, U^*, \psi_1, \ldots, \psi_l),$$

where $U^* = \lambda s.M^*$ and $SC(N, \psi_1, \ldots, \psi_l)$ stands for the set of functions over M^* *search computable with constants derived from* N *in* $\psi_1, \ldots, \psi_l$. (Cf. Skordev [1980] about this equality.)

Proposition 24.4 (Search Computability Theorem). Proposition 24.3 holds with SC substituted for PC and U^* added to the initial elements in (2), (3).

As shown in Soskov [1983], the functions over M prime computable in $U = L \cup R$ are in essence those computable by the effective definitional schemes of Friedman [1971] or, more precisely, by the recursively enumerable definitional schemes of Shepherdson [1975], hence st-recursiveness also subsumes the latter notions. Notice that while st-recursiveness subsumes the prime and search computability of Moschovakis and Friedman's computability, by 23.1, 23.4 so does recursiveness as far as functions over M rather than M^* are concerned.

In order to express the notion of hyperprojective function of Moschovakis [1969], one should consider consecutive spaces $\mathscr{S}, \mathscr{S}'$ constructed from example 24.1 by 19.11. The corresponding IOS-notion is that of $\mathscr{B}'$-recursiveness with $\mathscr{B}' = \{\overline{K_0^*} - \overline{K_3^*}, St, Ex\}$, where Ex is the functional E of Moschovakis,

$t \in Ex(\varphi)(s)$

iff $(t = 0 \,\&\, \forall r \varphi((r,s))\downarrow \,\&\, \exists r(0 \in \varphi((r,s)))) \vee (t = 1 \,\&\, \forall r \exists u \neq 0 (u \in \varphi((r,s))))$.

We shall not go into further details since an analogous characterization via $\mathscr{B}'$-recursiveness will be obtained in chapter 29.

EXERCISES TO CHAPTER 24

Exercise 24.1. Show that the function Z is st-recursive, where $Z(0) = \bar{0}(0)$, $Z(s) = \bar{1}(s)$, if $s \in M$, and $Z(s) = R^2(s)$ otherwise.

Hint. $Z = K_0^* St(K_2^*) K_1^* (\bar{0}, K_0^* St(K_3^*) K_1^* R)$.

Exercise 24.2. Show that the function Nn is st-recursive, where $Nn(m) = 0$ for all m, and $Nn(s) = 1$ otherwise.

Hint. $Nn = R[Z(L, \breve{\mathrm{I}}L, K_1^* St(K_2^*)K_1^*(R, \breve{\mathrm{I}}L))]$.

Exercise 24.3. Show that the function Seq is st-recursive, where $Seq(s) = 0$, if $s = \langle s_1, \ldots, s_n \rangle$ for certain $n, s_1, \ldots, s_n$, and $Seq(s) = 1$ otherwise.

Hint. $Seq = R[\psi]$, where

$$\begin{aligned}\psi = Z(&\breve{\mathrm{I}}L, \breve{\mathrm{I}}L, St(Z)K_6(K_1^* K_4 K_2^* L, K_6(\breve{\mathrm{I}}L, St(K_4)K_1^* \circ \\ &St(Z)K_6(\breve{\mathrm{I}}L, K_6(\breve{\mathrm{I}}L, St(K_1^* St(K_2^*)K_1^*)K_6(K_1^* R, \breve{\mathrm{I}}L)))))).\end{aligned}$$

Exercise 24.4. Show that the function Eq is st-recursive, where $Eq(s) = 0$, if $s = (n, n)$ for a certain n, and $Eq(s) = 1$ otherwise.

Hint. $Eq = K_0^* St(Seq)K_1^*(K_4 Nn, \breve{\mathrm{I}})$.

Remark. Nn, Seq, Eq make it easy to construct more sophisticated st-recursive elements such as ρ, ρ_1, ρ_2 in the proof of 24.1. For example,

$$\rho_1 = K_1^* St(K_0^* St(Seq)K_1^*)K_6(K_6 K_1^* St(L)K_1^*, O).$$

CHAPTER 25

Further examples

This chapter studies several 'less standard' first order examples of IOS involving intuitions connected with probabilistic nondeterminism and the reliable estimation of functions. They originate in examples 13, 18, 19 of Skordev [1980], chapter 3.

Proposition 25.1 (Example 25.1). Let M be an infinite set with a splitting scheme f_1, f_2. Take $\mathscr{F} = \{\varphi/\varphi: M^2 \to [0, \infty]\}$, $\varphi \leq \psi$ iff $\forall st(\varphi(s,t) \leq \psi(s,t))$,

$$\varphi\psi(s,t) = \sum_r \varphi(s,r)\psi(r,t) = \sup\left\{\sum_{r\in N} \varphi(s,r)\psi(r,t)/N \subseteq M \text{ is finite}\right\},$$

$$(\varphi,\psi)(f_1(s),t) = \varphi(s,t), (\varphi,\psi)(f_2(s),t) = \psi(s,t)$$

and $(\varphi,\psi)(s) = 0$ otherwise, $I(s,s) = 1$ and $I(s,t) = 0$ otherwise, $L = \lambda st.I(f_1(s),t)$ and $R = \lambda st.I(f_2(s),t)$. Then $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is a $(**)_0$, $(***)_0$-complete OS.

Proof. The quadruple $(\mathscr{F}, \Pi, \lambda\theta.L\theta, \lambda\theta.R\theta)$ is both a CCPS and a SCPS by 16.4, 16.10 since all the subsets of $[0,\infty]$ have least upper bounds.

The equalities $\varphi I = I\varphi = \varphi$ are immediate. Let us verify that multiplication is associative and right distributive with respect to Π.

$$\begin{aligned}(\varphi\psi)\chi(s,t) &= \sum_r \varphi\psi(s,r)\chi(r,t) = \sum_r\left(\sum_u \varphi(s,u)\psi(u,r)\right)\chi(r,t)\\ &= \sum_r\sum_u \varphi(s,u)\psi(u,r)\chi(r,t) = \sum_u\sum_r \varphi(s,u)\psi(u,r)\chi(r,t)\\ &= \sum_u \varphi(s,u)\sum_r \psi(u,r)\chi(r,t) = \sum_u \varphi(s,u)\psi\chi(u,t) = \varphi(\psi\chi)(s,t)\end{aligned}$$

for all s,t, hence $(\varphi\psi)\chi = \varphi(\psi\chi)$. It follows that

$$\begin{aligned}(\varphi,\psi)\chi(f_1(s),t) &= \sum_r (\varphi,\psi)(f_1(s),r)\chi(r,t) = \sum_r \varphi(s,r)\chi(r,t)\\ &= \varphi\chi(s,t) = (\varphi\chi,\psi\chi)(f_1(s),t)\end{aligned}$$

and similarly $(\varphi,\psi)\chi(f_2(s),t) = \psi\chi(s,t) = (\varphi\chi,\psi\chi)(f_2(s),t)$, while $(\varphi,\psi)\chi(s,t) = (\varphi\chi,\psi\chi)(s,t) = 0$ otherwise, hence $(\varphi,\psi)\chi = (\varphi\chi,\psi\chi)$.

Let $\mathscr{H}$ be a well ordered subset of $\mathscr{F}$ and $\varphi = \lambda st \sup\{\theta(s,t)/\theta\in\mathscr{H}\}$. Following Skordev [1980], we show that $\varphi\psi = \sup(\mathscr{H}\psi)$ and $\psi\varphi = \sup\psi\mathscr{H}$ for all ψ. It is immediate that $\mathscr{H}\psi \leq \varphi\psi$. Suppose that $\mathscr{H}\psi \leq \tau$. If $r_1, \ldots, r_m$

are distinct members of M and $\theta_1,\ldots,\theta_m \in \mathscr{H}$, there is a $\theta \in \mathscr{H}$ such that $\theta_1,\ldots,\theta_m \le \theta$; hence

$$\sum_{i=1}^{m} \theta_i(s, r_i)\psi(r_i, t) \le \sum_{i=1}^{m} \theta(s, r_i)\psi(r_i, t) \le \theta\psi(s, t) \le \tau(s, t).$$

This is valid for all $m, \theta_1, \ldots, \theta_m, r_1, \ldots, r_m$; hence $\sum_{i=1}^{m} \varphi(s, r_i)\psi(r_i, t) \le \tau(s, t)$ for all $m, r_1, \ldots, r_m$, which implies $\sum_r \varphi(s, r)\psi(r, t) \le \tau(s, t)$. Therefore, $\varphi\psi(s, t) \le \tau(s, t)$ for all s, t, hence $\varphi\psi \le \tau$. Therefore, $\varphi\psi = \sup(\mathscr{H}\psi)$ and similarly $\psi\varphi = \sup \psi\mathscr{H}$. It follows in particular that $\circ$ is monotonic. Finally, 19.1–19.3 imply that $\mathscr{S}$ is a $(**)_0$, $(***)_0$-complete OS, which completes the proof.

It is worth mentioning however that the equality $(\sup \mathscr{H})\psi = \sup(\mathscr{H}\psi)$ does not hold for arbitrary $\mathscr{H}$. Indeed, the element

$$U = \sup\{L, R\} = \lambda st.(L(s, t) + R(s, t))$$

does not satisfy condition (2) of exercise 7.8 since $U(I, I) \not\le I$. The equality $U(\varphi, \psi) = \lambda st.(\varphi(s, t) + \psi(s, t))$ holds instead.

Proposition 25.2. Let $\mathscr{S}$ be the IOS of example 25.1 or a subspace of it. Then $\langle\ \ \rangle, \Delta, [\ \ ]$ are characterized as follows.

$$\langle\varphi\rangle(f_2^n(f_1(s)), f_2^n(f_1(t))) = \varphi(s, t),$$
$$\Delta(\varphi, \psi)(f_2^n(f_1(s)), t) = \sum_{r_1, \ldots, r_n} \varphi(s, r_1)\psi(r_1, r_2)\ldots\psi(r_n, t)$$

and $\langle\varphi\rangle(s, t)$, $\Delta(\varphi, \psi)(s, t) = 0$ otherwise,

$$[\varphi](s, t) = \sum_{n, r_0, \ldots, r_{n-1} \in f_2(M), r_n} I(s, r_0)\varphi(f_2^{-1}(r_0), r_1)\ldots\varphi(f_2^{-1}(r_{n-1}), r_n)I(r_n, f_1(t)).$$

Proof. The proof is based on the fact that $\langle\varphi\rangle = \Delta(\varphi L, R)$, $\Delta(\varphi, \psi) = \sup_n \Gamma^n(O)$, where $\Gamma = \lambda\theta.(\varphi, \theta\psi)$, while $[\varphi] = \sup_n \Gamma^n(O)$, $\Gamma = \lambda\theta.(I, \varphi\theta)$. We present the case of iteration only.

We have

$$\Gamma^1(O)(s, t) = (I, O)(s, t) = \sum_{r_0} I(s, r_0)I(r_0, f_1(t)).$$

Suppose that $n \ge 1$ and $\Gamma^n(O)(s, t) = \sum_{i<n, r_0, \ldots, r_{i-1} \in f_2(M), r_i} I(s, r_0)\varphi(f_2^{-1}(r_0), r_1)\cdots \varphi(f_2^{-1}(r_{i-1}), r_i)I(r_i, f_1(t))$. Then

$$\begin{aligned}
\Gamma^{n+1}(O)(s, t) &= (I, \varphi\Gamma^n(O))(s, t) = I(s, f_1(t)) + \sum_{r_0 \in f_2(M)} I(s, r_0)\varphi\Gamma^n(O)(f_2^{-1}(r_0), t) \\
&= I(s, f_1(t)) + \sum_{r_0 \in f_2(M)} I(s, r_0)\sum_r \varphi(f_2^{-1}(r_0), r)\Gamma^n(O)(r, t) \\
&= \sum_{r_0} I(s, r_0)I(r_0, f_1(t)) + \sum_{r_0 \in f_2(M)} I(s, r_0)\sum_r \varphi(f_2^{-1}(r_0), r) \\
&\quad \times \sum_{0<i<n+1, r_1, \ldots, r_{i-1} \in f_2(M), r_i} I(r, r_1)\varphi(f_2^{-1}(r_1), r_2)\ldots\varphi(f_2^{-1}(r_{i-1}), r_i)I(r_i, f_1(t)) \\
&= \sum_{i<n+1, r_0, \ldots, r_{i-1} \in f_2(M), r_i} I(s, r_0)\varphi(f_2^{-1}(r_0), r_1)\ldots\varphi(f_2^{-1}(r_{i-1}), r_i)I(r_i, f_1(t)).
\end{aligned}$$

Using the equality $[\varphi](s, t) = \sup_n(\Gamma^n(O)(s, t))$, we get the desired characterization of $[\ \ ]$. This completes the proof.

Of course, the pairing space of example 25.1 can be augmented with multiplication in another way to become the IOS of example 21.1 with $E = [0, \infty]$. Propositions 21.2, 25.2 show that the corresponding probabilistic and fuzzy spaces have identical operations $\Pi, \langle\ \ \rangle$ but different $\circ, \Delta, [\ \ ]$.

The following space consists of *probabilistic functions.*

Proposition 25.3 (Example 25.2). Let $\mathscr{S}_0 = (\mathscr{F}_0, I, \Pi_0, L, R)$ be the IOS of example 25.1. Take

$$\mathscr{F} = \left\{ \varphi/\varphi : M^2 \to [0, 1] \,\&\, \forall s \left(\sum_t \varphi(s, t) \leq 1 \right) \right\}.$$

Then $\mathscr{S} = (\mathscr{F}, I, \Pi_0 \restriction \mathscr{F}^2, L, R)$ is a $(**)_0$, $(***)_0$-complete OS.

Proof. It is immediate that $I, L, R \in \mathscr{F}$ and $\mathscr{F}$ is closed under Π_0. If $\varphi, \psi \in \mathscr{F}$, then

$$\sum_t \varphi\psi(s, t) = \sum_t \sum_r \varphi(s, r)\psi(r, t) = \sum_r \varphi(s, r) \sum_t \psi(r, t) \leq \sum_r \varphi(s, r) \leq 1$$

for all s; hence $\varphi\psi \in \mathscr{F}$. Therefore, $\mathscr{F}$ is closed under $\circ$.

As shown in Skordev [1980], $\mathscr{F}$ is closed under least upper bounds of well ordered subsets. In fact, let $\mathscr{H}$ be such a subset of $\mathscr{F}$ and $\varphi = \sup \mathscr{H} = \lambda st. \sup \{\theta(s, t)/\theta \in \mathscr{H}\}$. Let $t_1, \ldots, t_m$ be distinct members of M and $\theta_1, \ldots, \theta_m \in \mathscr{H}$. Then there is a $\theta \in \mathscr{H}$ such that $\theta_1, \ldots, \theta_m \leq \theta$, hence

$$\sum_{i=1}^{m} \theta_i(s, t_i) \leq \sum_{i=1}^{m} \theta(s, t_i) \leq \sum_t \theta(s, t) \leq 1.$$

This holds for all $m, \theta_1, \ldots, \theta_m$, $t_1, \ldots, t_m$, hence $\sum_{i=1}^{m} \varphi(s, t_i) \leq 1$ for all m, $t_1, \ldots, t_m$, which implies $\sum_t \varphi(s, t) \leq 1$, hence $\varphi \in \mathscr{F}$. Therefore, $\mathscr{S}$ is a $(**)_0$, $(***)_0$-complete subspace of $\mathscr{S}_0$ by 18.15, 18.16. The proof is complete.

A few words about the intuition behind example 25.2. The members of $\mathscr{F}$ may be regarded as semantical counterparts of programs processed by a computer as in chapter 21; in this case $\varphi(s, t) \in [0, 1]$ is the probability of s being processed into t. This intuitive interpretation is extended to cover example 25.1 by regarding $\varphi(s, t) \in [0, \infty]$ as the average number of trajectories leading from s to t (from t to s, in another version); cf. Skordev [1980] for a further discussion.

Proposition 25.4 (Example 25.3). Let M be an infinite set with a splitting scheme f_1, f_2. Take $\mathscr{F} = \{\varphi/\varphi : M \to 2^M\}$, $\varphi \leq \psi$ iff ψ is an extension of φ, i.e. $\operatorname{Dom} \varphi \subseteq \operatorname{Dom} \psi$ and $\psi(s) = \varphi(s)$ for all $s \in \operatorname{Dom} \varphi$, where $\operatorname{Dom} \varphi = \{s/\varphi(s) \neq \varnothing\}$. Take $\varphi\psi(s) = \psi(\varphi(s))$, if $s \in \operatorname{Dom} \varphi$, $\varphi(s) \subseteq \operatorname{Dom} \psi$, and $\varphi\psi(s)\uparrow$ otherwise, $(\varphi, \psi)(f_1(s)) = \varphi(s)$, $(\varphi, \psi)(f_2(s)) = \psi(s)$ and $(\varphi, \psi)(s)\uparrow$ otherwise, $I = \lambda s.s$, $L = f_1$ and $R = f_2$. Then $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is a $(**)_0$-complete OS.

Proof. It easily follows that $\leq$ is a partial order, the operations $\circ, \Pi$ are monotonic and I is a unit.

$$\begin{aligned} \operatorname{Dom} (\varphi\psi)\chi &= \{s \in \operatorname{Dom} \varphi\psi/\varphi\psi(s) \subseteq \operatorname{Dom} \chi\} \\ &= \{s \in \operatorname{Dom} \varphi/\varphi(s) \subseteq \operatorname{Dom} \psi \,\&\, \psi(\varphi(s)) \subseteq \operatorname{Dom} \chi\} \\ &= \{s \in \operatorname{Dom} \varphi/\varphi(s) \subseteq \operatorname{Dom} \psi\chi\} = \operatorname{Dom} \varphi(\psi\chi) \end{aligned}$$

and if $s \in \mathrm{Dom}\,(\varphi\psi)\chi$, then

$$(\varphi\psi)\chi(s) = \chi(\psi(\varphi(s))) = \varphi(\psi\chi)(s);$$

hence $(\varphi\psi)\chi = \varphi(\psi\chi)$.

$$\begin{aligned}\mathrm{Dom}\,(\varphi, \psi)\chi &= \{s \in \mathrm{Dom}\,(\varphi, \psi)/(\varphi, \psi)(s) \subseteq \mathrm{Dom}\,\chi\}\\ &= \{f_1(s)/s \in \mathrm{Dom}\,\varphi \,\&\, \varphi(s) \subseteq \mathrm{Dom}\,\chi\}\\ &\quad \cup \{f_2(s)/s \in \mathrm{Dom}\,\psi \,\&\, \psi(s) \subseteq \mathrm{Dom}\,\chi\}\\ &= \{f_1(s)/s \in \mathrm{Dom}\,\varphi\chi\} \cup \{f_2(s)/s \in \mathrm{Dom}\,\psi\chi\} = \mathrm{Dom}\,(\varphi\chi, \psi\chi)\end{aligned}$$

and if $f_1(s) \in \mathrm{Dom}(\varphi, \psi)\chi$, then $(\varphi, \psi)\chi(f_1(s)) = \varphi\chi(s)$, respectively $(\varphi, \psi)\chi(f_2(s)) = \psi\chi(s)$, if $f_2(s) \in \mathrm{Dom}\,(\varphi, \psi)\chi$; hence $(\varphi, \psi)\chi = (\varphi\chi, \psi\chi)$.

$$\mathrm{Dom}\, L(\varphi, \psi) = \{s/f_1(s) \in \mathrm{Dom}\,(\varphi, \psi)\} = \mathrm{Dom}\,\varphi$$

and if $s \in \mathrm{Dom}\,\varphi$, then

$$L(\varphi, \psi)(s) = (\varphi, \psi)(f_1(s)) = \varphi(s),$$

hence $L(\varphi, \psi) = \varphi$. Similarly, $R(\varphi, \psi) = \psi$. Therefore, $\mathscr{S}$ is an OS.

Let $\mathscr{H}$ be a well ordered subset of $\mathscr{F}$ and $\psi \in \mathscr{F}$. Take $\varphi = \cup\mathscr{H}$. Then $\theta \leq \varphi$ for all $\theta \in \mathscr{H}$ since $\mathscr{H}$ is well ordered, hence $\mathscr{H}\psi \leq \varphi\psi$. Suppose that $\mathscr{H}\psi \leq \tau$. If $s \in \mathrm{Dom}\,\varphi\psi$, then $s \in \mathrm{Dom}\,\varphi$, hence there is a $\theta \in \mathscr{H}$ such that $s \in \mathrm{Dom}\,\theta$ and $\varphi(s) = \theta(s)$. It follows that $s \in \mathrm{Dom}\,\theta\psi$; hence $s \in \mathrm{Dom}\,\tau$ and $\tau(s) = \theta\psi(s) = \varphi\psi(s)$. This holds for all $s \in \mathrm{Dom}\,\varphi\psi$; hence $\varphi\psi \leq \tau$. Therefore, $\varphi\psi = \sup(\mathscr{H}\psi)$.

Let $\mathscr{H}$ and φ be the same as above. Then $L\mathscr{H} \leq L\varphi$. Suppose that $L\mathscr{H} \leq \tau$. If $s \in \mathrm{Dom}\, L\varphi$, then $f_1(s) \in \mathrm{Dom}\,\varphi$ and $L\varphi(s) = \varphi(f_1(s))$. There is a $\theta \in \mathscr{H}$ such that $f_1(s) \in \mathrm{Dom}\,\theta$ and $\theta(f_1(s)) = \varphi(f_1(s))$; hence $\tau(s) = L\theta(s) = L\varphi(s)$. Therefore, $L\,\varphi \leq \tau$; hence $L\,\varphi = \sup L\,\mathscr{H}$. Similarly, $R\varphi = \sup R\mathscr{H}$; hence $\mathscr{S}$ is $(**)_0$ complete. The proof is complete.

Proposition 25.5. The operations $\langle\ \ \rangle, \Delta, [\ \ ]$ of example 25.3 can be characterized as follows.

$$\begin{aligned}\langle\varphi\rangle(f_2^n(f_1(s))) &= f_2^n(f_1(\varphi(s))),\\ \Delta(\varphi, \psi)(f_2^n(f_1(s))) &= \varphi\psi^n(s),\end{aligned}$$

and $\langle\varphi\rangle(s)$, $\Delta(\varphi, \psi)(s)\uparrow$ otherwise.

We say that t is a *φ-successor of* $f_2(s)$ iff $t \in \varphi(s)$, while s is *φ-regular* iff $s \in f_1(M) \cup f_2(\mathrm{Dom}\,\varphi)$. Take the least subset D_φ of M such that whenever s is φ-regular and all its φ-successors are in D_φ, then $s \in D_\varphi$. Then $t \in [\varphi](s)$ iff $s \in D_\varphi$ and there are $n, r_0, \ldots, r_n$ such that $r_0 = s$, r_{i+1} is a φ-successor of r_i for all $i < n$, and $r_n = f_1(t)$.

Proof. We take $\Gamma = \lambda\theta.(\varphi, \theta\psi)$ and prove by induction on n that $\Gamma^n(O)(f_2^i(f_1(s))) = \varphi\psi^i(s)$, if $i < n$, and $\Gamma^n(O)(s)\uparrow$ otherwise. Exercise 18.4 gives $\Delta(\varphi, \psi) = \sup_n \Gamma^n(O)$, which implies the desired characterization of Δ and that of $\langle\ \ \rangle$ by $\langle\varphi\rangle = \Delta(\varphi L, R)$.

Let $t \in \sigma(s)$ iff $s \in D_\varphi$ and there are $n, r_0, \ldots, r_n$ such that $r_0 = s$, r_{i+1} is a φ-successor of r_i for all $i < n$, and $r_n = f_1(t)$. We show that $[\varphi] = \sigma$, following a similar argument from example 18, Skordev [1980], chapter 3.

Let us show first that $\operatorname{Dom}\sigma = D_\varphi$. In order to get $D_\varphi \subseteq \operatorname{Dom}\sigma$ it suffices to show that whenever s is φ-regular and all its φ-successors are in $\operatorname{Dom}\sigma$, then $s \in \operatorname{Dom}\sigma$. If $s = f_1(t)$, then $s \in D_\varphi$ and $\sigma(s) = t$, hence $s \in \operatorname{Dom}\sigma$. Let $s = f_2(r)$, $r \in \operatorname{Dom}\varphi$. Take a $t \in \varphi(r)$. Then t is a φ-successor of s, hence $t \in \operatorname{Dom}\sigma$. All φ-successors of s are in $\operatorname{Dom}\sigma \subseteq D_\varphi$, hence $s \in D_\varphi$. It follows that $\sigma(t) \subseteq \sigma(s)$, hence $\sigma(s) \neq \varnothing$ and $s \in \operatorname{Dom}\sigma$.

Our next objective is to prove that $(I, \varphi\sigma) \leq \sigma$. Suppose that $s \in \operatorname{Dom}(I, \varphi\sigma)$ and write N for $(I, \varphi\sigma)(s)$. In other words, either $s \in f_1(M)$ or $s \in f_2(\operatorname{Dom}\varphi)$ and $\varphi(f_2^{-1}(s)) \subseteq D_\varphi$, while

$$N = \{t/s = f_1(t) \vee t \in \sigma(\varphi(f_2^{-1}(s)))\}.$$

Then s is φ-regular and all its φ-successors are in D_φ; hence $s \in D_\varphi = \operatorname{Dom}\sigma$. It follows easily that $N \subseteq \sigma(s)$. Conversely, suppose that $t \in \sigma(s)$. Then there are $n, r_0, \ldots, r_n$ such that $r_0 = s$, r_{i+1} is a φ-successor of r_i for all $i < n$, and $r_n = f_1(t)$. If $n = 0$, then $s = f_1(t)$; hence $t \in N$. If $n > 0$, then $s \in f_2(M)$ and $t \in \sigma(\varphi(f_2^{-1}(s)))$, hence $t \in N$ again. We get $\sigma(s) \subseteq N$; hence $\sigma(s) = (I, \varphi\sigma)(s)$ for all $s \in \operatorname{Dom}(I, \varphi\sigma)$, which implies $(I, \varphi\sigma) \leq \sigma$.

Suppose that $(I, \varphi\tau) \leq \tau$. Then for all s

(1) $s \in f_1(M) \vee s \in f_2(\operatorname{Dom}\varphi) \,\&\, \varphi(f_2^{-1}(s)) \subseteq \operatorname{Dom}\tau \Rightarrow s \in \operatorname{Dom}\tau \,\&\, \tau(s)$
$= \{t/s = f_1(t) \vee t \in \tau(\varphi(f_2^{-1}(s)))\}$.

If s is φ-regular and all its φ-successors are in $\operatorname{Dom}\tau$, then $s \in \operatorname{Dom}\tau$ by (1); hence $D_\varphi \subseteq \operatorname{Dom}\tau$.

Let $N = \{s \in D_\varphi / \tau(s) = \sigma(s)\}$. Whenever s is φ-regular and all its φ-successors are in N, then all its φ-successors are in D_φ; hence $s \in D_\varphi$. Moreover, all φ-successors of s are in $\operatorname{Dom}\tau$ since $D_\varphi \subseteq \operatorname{Dom}\tau$, so (1) gives

$$\tau(s) = \{t/s = f_1(t) \vee t \in \sigma(\varphi(f_2^{-1}(s)))\};$$

hence $\tau(s) = (I, \varphi\sigma)(s) = \sigma(s)$ and $s \in N$. The definition of D_φ implies $D_\varphi \subseteq N$, hence $\tau(s) = \sigma(s)$ for all $s \in D_\varphi$. Therefore, $\sigma \leq \tau$.

We conclude that $\sigma = \mu\theta.(I, \varphi\theta)$, hence $[\varphi] = \sigma$. The proof is complete.

The spaces of examples 22.1, 25.3 have the same carriers $\mathscr{F}$ and operations $\Pi, \langle\ \ \rangle$, but different $\leq, \circ, \Delta, [\ \ ]$.

Example 25.3 is related to the idea of reliable estimation of functions. Following Skordev [1980], an element $\varphi \in \mathscr{F}$ is a *reliable estimate* of a function $f: M \dashrightarrow M$ iff $\operatorname{Dom}\varphi \subseteq \operatorname{Dom} f$ and $f(s) \in \varphi(s)$ for all $s \in \operatorname{Dom}\varphi$. Given two reliable estimates φ, ψ of f, ψ is *better than* φ iff $\operatorname{Dom}\varphi \subseteq \operatorname{Dom}\psi$ and $\psi(s) \subseteq \varphi(s)$ for all $s \in \operatorname{Dom}\varphi$.

The IOS of example 22.2 is based on $\mathscr{F}_0 = \{f/f: M \dashrightarrow M\}$. Thus the members of $\mathscr{F}$ are reliable estimates of members of $\mathscr{F}_0$ and, as observed in the cited work, if φ, ψ are reliable estimates of f, g, then $\varphi\psi$ is a reliable estimate of fg. It will also be shown in the exercises that $(\varphi, \psi), \langle\varphi\rangle, [\varphi]$ are reliable estimates of $(f, g), \langle f\rangle$ and $[f]$.

As far as reliable estimates are concerned, the relation 'is better than' seems more natural than 'is an extension of'. Modifying $\mathscr{S}$ in this way, one gets the following example.

Proposition 25.6 (Example 25.4). Let $\mathscr{S}=(\mathscr{F},I,\Pi,L,R)$ be the OS of example 25.3 and $\mathscr{F}_1$ be obtained from $\mathscr{F}$ by substituting $\leq_1$ for $\leq$, where $\varphi\leq_1\psi$ iff ψ is better than φ. Then $\mathscr{S}_1=(\mathscr{F}_1,I,\Pi,L,R)$ is a μA_3-iterative OS and $\mathscr{S},\mathscr{S}_1$ have identical operations $\langle\quad\rangle$, $[\quad]$.

Proof. It follows easily that $\leq_1$ is a partial ordering of $\mathscr{F}$ and $\circ,\Pi$ are monotonic with respect to it; hence $\mathscr{S}_1$ is an OS. Suppose for instance that $\varphi\leq_1\varphi_1$, $\psi\leq_1\psi_1$. Then

$$\operatorname{Dom}\varphi\psi=\operatorname{Dom}\varphi\cap\varphi^{-1}(\operatorname{Dom}\psi)\subseteq\operatorname{Dom}\varphi_1\cap\varphi_1^{-1}(\operatorname{Dom}\psi_1)=\operatorname{Dom}\varphi_1\psi_1$$

and if $s\in\operatorname{Dom}\varphi\psi$, then

$$\varphi_1\psi_1(s)=\psi_1(\varphi_1(s))\subseteq\psi_1(\varphi(s))\subseteq\psi(\varphi(s))=\varphi\psi(s);$$

hence $\varphi\psi\leq_1\varphi_1\psi_1$.

Let $\mathscr{H}$ be a subset of $\mathscr{F}$ well ordered with respect to $\leq$ and let $\varphi=\sup\mathscr{H}$. Then $\varphi=\cup\mathscr{H}$ by the proof of 25.4. It is immediate that $\mathscr{H}\leq_1\varphi$. Suppose that $\mathscr{H}\leq_1\tau$. If $s\in\operatorname{Dom}\varphi$, then there is a $\theta\in\mathscr{H}$ such that $s\in\operatorname{Dom}\theta$ and $\theta(s)=\varphi(s)$. The inequality $\theta\leq_1\tau$ implies $s\in\operatorname{Dom}\tau$ and $\tau(s)\subseteq\theta(s)$; hence $\tau(s)\subseteq\varphi(s)$. This holds for all $s\in\operatorname{Dom}\varphi$; hence $\varphi\leq_1\tau$. Therefore, $\varphi=\sup_1\mathscr{H}$. It follows from 18.18 that $\mathscr{S}_1$ is μA_3-iterative and has the same initial operations as $\mathscr{S}$. The proof is complete.

Other interesting first order IOS-examples can be constructed corresponding to related examples from Skordev's book, bearing in mind the general connection between operative and combinatory spaces to be established in chapter 27. Among them are probabilistic spaces using countably additive measures on σ-algebras of sets rather than discrete probabilities, spaces connected with the notion of $\forall$-definiteness and topological versions of examples 22.1, 22.2, 25.3, 25.4. Some of those examples can be modified to accommodate the complexity measure of data processing.

EXERCISES TO CHAPTER 25

Exercise 25.1. Let $\mathscr{S}$ be the IOS of example 25.1 and $\varphi^{-1}=\lambda st.\varphi(t,s)$. Show that $\langle\!\langle\quad\rangle\!\rangle=\lambda\varphi.(\varphi,\varphi^{-1})$ is a t-operation with a corresponding set $\mathscr{B}_0=\{U\}$, and $\mathscr{S},\langle\!\langle\quad\rangle\!\rangle$ satisfy the axiom $t\mu A_3$.

Hint. Adapt the proof of 21.12 with $+$ substituted for sup, where $\varphi+\psi=\lambda st.(\varphi(s,t)+\psi(s,t))$.

Exercise 25.2. Let $\mathscr{S}$ be the IOS of example 25.1 or 25.2 and let J be the same as in 21.13. Take $St(\varphi)(J(s,t),\ J(s,r))=\varphi(t,r)$ and $St(\varphi)(s,t)=0$ otherwise. Show that St is a t-operation satisfying $t\mu A_3$.

Hint. Follow the proof of 21.13.

Analogue to 21.14 and the relevant exercises to chapter 21 may also be established.

Exercise 25.3. Let $\mathscr{S}$ be the IOS of example 25.3 or 25.4 and let St be introduced as in 21.13. Show that St is a t-operation satisfying $t\mu A_3$.

Hint. Follow the proof of 21.13. Use 18.21, respectively a t-analogue to 18.18.

Exercise 25.4. Show that, unlike translation, iteration in examples 25.3, 25.4 is not always reached at level ω as a least fixed point.

Hint. Modifying a counterexample of Skordev [1980], take $s_0 \in f_1(M)$, $s_{n+1} = f_1(s_n)$, if $\exists m(n+1 = 2^{m+1})$, and $s_{n+1} = f_2(s_n)$ otherwise, $\varphi(s_0) = \{s_2, s_3, \ldots\}$ and $\varphi(s_{n+1}) = s_{n+3}$, then take $\Gamma = \lambda\theta.(I, \varphi\theta)$, $\sigma = \sup_n \Gamma^n(O)$. Show that $s_1 \notin \mathrm{Dom}\,\sigma$ and $s_{n+2} \in \mathrm{Dom}\,\sigma$ for all n; hence $(I, \varphi\sigma) \neq \sigma$.

The following exercise shows however that bounded multiple-valuedness ensures ω-accessibility.

Exercise 25.5 (Examples 25.5, 25.6). Prove that the functions φ such that $\varphi(s)$ is finite for all s form iterative subspaces of examples 25.3 and 25.4 in which the least fixed points of all inductive mappings are reached at level ω.

Exercise 25.6. Let $\mathscr{S}_0, \mathscr{S}, \mathscr{S}_1$ be the IOS of examples 22.2, 25.3, 25.4 based on the same M, f_1, f_2. Show that whenever $\varphi, \psi \in \mathscr{F} = \mathscr{F}_1$ are reliable estimates respectively of $f, g \in \mathscr{F}_0$, then (φ, ψ), $\langle \varphi \rangle$, $[\varphi]$ are reliable estimates of (f, g), $\langle f \rangle$, $[f]$.

Hint. Use 18.15 to show that $\mathscr{S}_0$ is a subspace of $\mathscr{S}$, which implies that $\mathscr{S}_0$ is a subspace of $\mathscr{S}_1$. Use the fact that φ is a reliable estimate of f iff $\varphi \leq_1 f$.

Remark. It follows that whenever all the members of $\mathscr{B}_0 \subseteq \mathscr{F}_0$ have reliable estimates in $\mathscr{B} \subseteq \mathscr{F} = \mathscr{F}_1$ and f is recursive in $\mathscr{B}_0$, then f has a reliable estimate φ recursive in $\mathscr{B}$.

Exercise 25.7 (Example 25.7). Let M be an infinite set with a splitting scheme f_1, f_2 and $u \notin M$. Take $\mathscr{F} = \{\varphi / \varphi : M \cup \{u\} \to 2^{M \cup \{u\}} \,\&\, \varphi(u) = u\}$, $\varphi \leq \psi$ iff $\forall s(\varphi(s) \subseteq \psi(s))$, $\varphi\psi = \lambda s.\psi(\varphi(s))$, $(\varphi, \psi)(f_1(s)) = \varphi(s)$, $(\varphi, \psi)(f_2(s)) = \psi(s)$ and $(\varphi, \psi)(s) = u$ otherwise, $I = \lambda s.s$, and extend f_1, f_2 to $L, R \in \mathscr{F}$. Show that $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is a $\mu\mathrm{A}_3$-iterative OS. Give explicit characterizations of $\langle \quad \rangle$, $[\quad]$.

Remark. This modification of example 22.1 is an analogue to example 2 of Skordev [1980a]. Example 22.2 can be modified to form a subspace again. Intuitively, u indicates that an unproductive termination has occurred. Notice that $\varphi O = O$ fails, and if $f_1(M) \cup f_2(M) \subset M$, then $(O, O) \neq O$.

Exercise 25.8. Let $\mathscr{S}$ be the space of example 25.7, let J be as in 21.13 and define St as follows. If $r \in \varphi(t)$, $r \neq u$, take $J(s, r) \in St(\varphi)(J(s, t))$. If $u \in \varphi(t)$, take $u \in St(\varphi)(J(s, t))$. Take $St(\varphi)(s) = u$, provided $s \notin J(M_0, M)$. Show that St is a t-operation satisfying $\mathrm{t}\mu\mathrm{A}_3$.

Exercise 25.9. Show that the element $V = \lambda s.\{f_1(s), f_2(s)\}$ in example 25.4 satisfies condition (1) of exercise 7.14, while in example 25.7 $U = L \cup R$ satisfies condition (1) of exercise 7.10.