PART B

Pure recursion theory on operative spaces

CHAPTER 4

Operative spaces

The algebraic system of operative space is introduced in this chapter and its operations of multiplication and pairing are studied.

Suppose we are given a partially ordered semigroup $\mathscr{F}$ with a unit I, an operation $\Pi:\mathscr{F}^2 \to \mathscr{F}$ called *pairing* and two distinct elements $L, R \in \mathscr{F}$. Some notation: small greek letters range over $\mathscr{F}$, while certain capital latin letters stand for fixed elements of $\mathscr{F}$. For the sake of brevity we write (φ, ψ) for $\Pi(\varphi, \psi)$ and $(\varphi_1, \ldots, \varphi_n)$ for $(\varphi_1, (\varphi_2, \ldots, \varphi_n))$, provided $n > 2$. Occasionally $\circ$ denotes the semigroup *multiplication* of $\mathscr{F}$. Natural numbers are represented in $\mathscr{F}$ by elements of the form $\bar{n} = LR^n$.

The 5-tuple $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is said to be an *operative space* (OS for short) iff the following three axioms are also satisfied.

$$\begin{aligned} &\mathrm{A}_1. \quad \varphi \leq \varphi_1, \psi \leq \psi_1 \Rightarrow (\varphi, \psi) \leq (\varphi_1, \psi_1). \\ &\mathrm{A}_2. \quad (\varphi, \psi)\chi = (\varphi\chi, \psi\chi). \\ &\mathrm{A}_3. \quad L(\varphi, \psi) = \varphi, \; R(\varphi, \psi) = \psi. \end{aligned}$$

A *subspace* of $\mathscr{S}$ is a 5-tuple $\mathscr{S}_1 = (\mathscr{F}_1, I, \Pi \upharpoonright \mathscr{F}_1^2, L, R)$ such that $\mathscr{F}_1 \subseteq \mathscr{F}$, $I, L, R \in \mathscr{F}_1$ and $\mathscr{F}_1$ is closed under the operations $\circ, \Pi$. Of course, any subspace of $\mathscr{S}$ is itself an OS.

We give several examples of OS to illustrate the above definitions.

Example 4.1. The OS $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ of example 3.1.

Example 4.2. The subspace $\mathscr{S}_1 = (\mathscr{F}_1, I, \Pi \upharpoonright \mathscr{F}_1^2, L, R)$ of the preceding OS, based on $\mathscr{F}_1 = \{\varphi/\varphi:\omega \to \omega\}$. In other words, $\mathscr{F}_1$ consists of those functions in $\mathscr{F}$ which are total.

Example 4.3. Example 4.1 with a *pairing scheme* Π, L, R modified as follows: Take $(\varphi, \psi)(3s) = \varphi(s)$, $(\varphi, \psi)(3s+1) = \psi(s)$, $(\varphi, \psi)(3s+2)\uparrow$, $L = \lambda s.3s$ and $R = \lambda s.3s+1$.

Example 4.4. Let M be an arbitrary infinite set, let $J:M^2 \to M$ be injective (such a *pairing function* J exists since M is infinite), and let $L, R:M \to M$ satisfy $L(J(s,t)) = s$, $R(J(s,t)) = t$ for all $s, t \in M$. Then take $\mathscr{F} = \{\varphi/\varphi:M \to M\}$, $I = \lambda s.s$, $\varphi \leq \psi$ iff $\varphi = \psi$, $\varphi\psi = \lambda s.\varphi(\psi(s))$ and $(\varphi, \psi) = \lambda s.J(\varphi(s), \psi(s))$.

Example 4.5. Example 4.4 with $M=\omega$; in particular, one may use the pairing functions $J=\lambda st.2^s(2t+1)-1$, $\lambda st.(2s+1)2^t$, $\lambda st.\frac{1}{2}(s+t)(s+t+1)+s$ (Cantor's function) or $\lambda st.2^s3^t$.

Example 4.6. The OS of example 3.2.

Assume from now on that on OS $\mathscr{S}$ is given. Let $\mathscr{D}=cl(I,L,R/\circ)$ and α,β range over $\mathscr{D}$.

Proposition 4.1. $\mathscr{D}=cl(I/\lambda\theta.\theta L,\lambda\theta.\theta R)$.

Proof. Write $\mathscr{D}_1$ for $cl(I/\lambda\theta.\theta L,\lambda\theta.\theta R)$. The inclusion $\mathscr{D}_1\subseteq\mathscr{D}$ is obvious. In order to get $\mathscr{D}_1=\mathscr{D}$ it suffices to show that whenever $\alpha,\beta\in\mathscr{D}_1$, then $\alpha\beta\in\mathscr{D}_1$. This can be done by induction on the construction of β. If $\beta=I$ and $\alpha\in\mathscr{D}_1$, then $\alpha\beta=\alpha\in\mathscr{D}_1$. Suppose that $\beta=\beta_1L$ and $\alpha\beta_1\in\mathscr{D}_1$ for all $\alpha\in\mathscr{D}_1$. Then $\alpha\beta_1L\in\mathscr{D}_1$, i.e. $\alpha\beta\in\mathscr{D}_1$ for all $\alpha\in\mathscr{D}_1$. The case $\beta=\beta_1R$ is treated similarly.

Proposition 4.2. For every α there is an n such that $\alpha(I,I)^n=I$.

Proof. If $\alpha=I$, then $\alpha(I,I)^0=\alpha I=I$.

Let $\alpha=\alpha_1L$ or $\alpha=\alpha_1R$ and suppose that the assertion holds for α_1. Then there is a n such that $\alpha_1(I,I)^n=I$, hence $\alpha(I,I)^{n+1}=\alpha_1(I,I)^n=I$, which completes the proof by 4.1.

Proposition 4.3. $\alpha L\nleq\beta R$, $\alpha R\nleq\beta L$ (for all α,β: we often omit these words).

Proof. Suppose $\alpha L\leq\beta R$. Proposition 4.2 implies that there are φ,ψ such that $\alpha\varphi=\beta\psi=I$. Multiplying the inequality $\alpha L\leq\beta R$ by $(\varphi L,\psi R)$, we get $L\leq R$. Multiplying the last inequality by (R,L), we get $R\leq L$, hence $L=R$, which is not the case.

Supposing $\alpha R\leq\beta L$ and multiplying by (R,L), we get $\alpha L\leq\beta R$ contrary to the assertion just proved. This completes the proof.

Consider the alphabet $\mathbb{L}\mathbb{R}$. To each word a in it we assign an element α. Namely, say that the empty word Λ *represents* I and whenever a represents α, then $a\mathbb{L}$, $a\mathbb{R}$ represent respectively αL, αR.

Proposition 4.4. Let a,b represent respectively α,β and $\alpha\leq\beta$. Then $a=b$ (a and b are graphically identical), which in particular implies $\alpha=\beta$.

Proof. By induction on the construction of a.

Let $a=\Lambda$. Suppose that $b=b_1\mathbb{L}$. Then $\beta=\beta_1L$, where b_1 represents β_1. Multiplying the inequality $I\leq\beta_1L$ on the left by R, we get $R\leq R\beta_1L$ contrary to 4.3. The possibility $b=b_1\mathbb{R}$ is also ruled out. Therefore, $b=\Lambda=a$.

Let $a=a_1\mathbb{L}$ and a_1 meet the induction hypothesis. Supposing $b=\Lambda$, we get a contradiction as above. Suppose that $b=b_1\mathbb{R}$. Then $\alpha=\alpha_1L$ and $\beta=\beta_1R$, where a_1,b_1 represent respectively α_1,β_1. We get $\alpha_1L\leq\beta_1R$ contrary to 4.3. Therefore, $b=b_1\mathbb{L}$, hence $\alpha=\alpha_1L$, $\beta=\beta_1L$ with a_1,b_1 representing respectively α_1,β_1. The inequality $\alpha_1L\leq\beta_1L$ multiplied by (I,I) gives $\alpha_1\leq\beta_1$, which implies $a_1=b_1$ by the induction hypothesis. Therefore, $a=b$. The case of $a=a_1\mathbb{R}$ is treated similarly.

Proposition 4.5. If $\exists\chi(\chi\varphi\in\mathscr{D})$, then $\varphi L \neq \psi R$ and $\varphi R \neq \psi L$ for all ψ.

Proof. Let $\chi\varphi = \alpha$. Then $\varphi L = \psi R$ implies $\alpha L = \chi\psi R$. Multiplying by (R, R), we get $\alpha R = \chi\psi R$, hence $\alpha L = \alpha R$ contrary to 4.3.

The equality $\varphi R = \psi L$ multiplied by (R, L) gives $\varphi L = \psi R$, which is not the case. Thus the proof is complete.

It follows by 4.5 that $\alpha L \neq \psi R$ and $\alpha R \neq \psi L$ for all α, ψ.

Proposition 4.6. $\varphi L \neq I$, $\varphi R \neq I$.

Proof. The equality $\varphi L = I$ multiplied on the left by R gives $R = R\varphi L$ contrary to 4.5. Similarly $\varphi R \neq I$.

Proposition 4.7. Whenever $\varphi\alpha\in\mathscr{D}$, then $\varphi\in\mathscr{D}$.

Proof. The case $\alpha = I$ is immediate. Let $\alpha = \alpha_1 L$ and whenever $\varphi\alpha_1\in\mathscr{D}$, then $\varphi\in\mathscr{D}$. Let $\varphi\alpha = \beta$. Then $\varphi\alpha_1 L = \beta$, hence $\beta = \beta_1 L$ for a certain β_1 by 4.5, 4.6. Multiplying by (I, I), we get $\varphi\alpha_1 = \beta_1$, hence $\varphi\in\mathscr{D}$. The case of $\alpha = \alpha_1 R$ is treated similarly.

Proposition 4.8. $(I, I)\notin\mathscr{D}$. Therefore, $\mathscr{D}$ is not closed under Π.

Proof. Suppose that $(I, I) = \alpha$. Then $L\alpha = L(I, I) = I$ contrary to 4.4.

Proposition 4.9. If $(\varphi, \psi)\in\mathscr{D}$, then $\varphi, \psi\in\mathscr{D}$.

Proof. If $(\varphi, \psi) = \alpha$, then $\varphi = L\alpha$ and $\psi = R\alpha$, hence $\varphi, \psi\in\mathscr{D}$.

Moreover, it follows that

$$\alpha = (\varphi, \psi) = (L\alpha, R\alpha) = (L, R)\alpha,$$

hence $(L, R) = I$ by 4.2. It is worth mentioning however that the last equality fails in some OS, e.g. example 4.3. Therefore, $(\varphi, \psi)\notin\mathscr{D}$ for all φ, ψ in such cases.

Proposition 4.10. If $\bar{m} \leq \bar{n}$, then $m = n$.

This follows from 4.4. Consequently, our way of representing natural numbers in $\mathscr{F}$ is correct.

Proposition 4.11. Let $n \geq 1$. Then $\bar{k}(\varphi_0, \ldots, \varphi_n) = \varphi_k$ for all $k < n$, while $R^n(\varphi_0, \ldots, \varphi_n) = \varphi_n$.

This is proved by an easy induction on n.

Proposition 4.12. Let $\varphi\in cl(I, L, R/\circ, \Pi)$. Then there is a natural number m and elements $\varphi_0, \ldots, \varphi_m$ such that $\varphi_m = \varphi$, for all $i \leq m$ either $\varphi_i\in\mathscr{D}$ or $\varphi_i = (\varphi_k, \varphi_l)$ with $k, l < i$, and for all α either $\alpha\varphi\in\mathscr{D}$ or $\alpha\varphi = \varphi_i$ for a certain $i \leq m$.

Proof. If $\varphi\in\{I, L, R\}$, then take $m = 0$, $\varphi_0 = \varphi$.

Let $\varphi_0, \ldots, \varphi_m$ and $\psi_0, \ldots, \psi_n$ correspond to φ, ψ respectively.

Let $\chi = (\varphi, \psi)$. Then take $k = m + n + 2$, $\chi_i = \varphi_i$ for $i \leq m$, $\chi_{m+i+1} = \psi_i$ for $i \leq n$ and $\chi_{m+n+2} = \chi$. Let $\alpha\in\mathscr{D}$. If $\alpha = I$, then $\alpha\chi = \chi = \chi_{m+n+2}$. If $\alpha = \alpha_1 L$, then $\alpha\chi = \alpha_1\varphi$ and the induction assumption for φ applies. If $\alpha = \alpha_1 R$, then $\alpha\chi = \alpha_1\psi$ and we use the assumption for ψ.

Let $\chi = \varphi\psi$. We prove by induction on $i, i \leq m$, that $\varphi_i\psi$ has the required properties. If $\varphi_i \in \mathscr{D}$, then the assumption for ψ yields that either $\varphi_i\psi \in \mathscr{D}$ or $\varphi_i\psi = \psi_j$ with a certain $j \leq n$, whence $\varphi_i\psi$ has the required properties. If $\varphi_i = (\varphi_k, \varphi_l)$ for some $k, l < i$, then $\varphi_i\psi = (\varphi_k\psi, \varphi_l\psi)$. The assertion holds for both $\varphi_k\psi$, $\varphi_l\psi$ and we proceed as in the case of $\chi = (\varphi, \psi)$. This completes the proof.

Proposition 4.12 reflects the intuitively clear fact that $cl(I, L, R/\circ, \Pi) = cl(\mathscr{D}/\Pi)$. If $\{I, L, R\}$ is replaced by an arbitrary subset $\mathscr{B}$ of $\mathscr{F}$, then something weaker but still useful can be proved.

Proposition 4.13.

$$cl(\mathscr{B}/\circ, \Pi) = cl(\mathscr{B}/\Pi, \lambda\theta.\psi\theta \text{ for all } \psi \in \mathscr{B}).$$

Proof. Let $\mathscr{B}^* = cl(\mathscr{B}/\Pi, \lambda\theta.\psi\theta$ for all $\psi \in \mathscr{B})$. It suffices to show that whenever $\varphi \in cl(\mathscr{B}/\circ, \Pi)$, then $\varphi \in \mathscr{B}^*$ and $\varphi\psi \in \mathscr{B}^*$ for all $\psi \in \mathscr{B}^*$.

Certainly, all the members of $\mathscr{B}$ have this property.

Suppose that φ_1, φ_2 have the property in question. Then $\varphi_1, \varphi_2 \in \mathscr{B}^*$; hence $\varphi_1\varphi_2 \in \mathscr{B}^*$ and $(\varphi_1, \varphi_2) \in \mathscr{B}^*$. Let $\psi \in \mathscr{B}^*$. Then $\varphi_2\psi \in \mathscr{B}^*$, hence $\varphi_1\varphi_2\psi \in \mathscr{B}^*$. It also follows that $\varphi_1\psi \in \mathscr{B}^*$, hence $(\varphi_1, \varphi_2)\psi = (\varphi_1\psi, \varphi_2\psi) \in \mathscr{B}^*$. Therefore, both $\varphi_1\varphi_2$ and (φ_1, φ_2) have the required property, which completes the proof.

EXERCISES TO CHAPTER 4

The first two exercises introduce new examples of OS which correspond to examples 1, 2 in Skordev [1980], chapter 2.

Exercise 4.1 (Example 4.7). Let M be an infinite set, $L, R: M \to M$ be injective and $L(M) \cap R(M) = \varnothing$. (Such L, R exist since M is infinite.) Take $\mathscr{F} = \{\varphi/\varphi: M \dashrightarrow M\}$, $I = \lambda s.s$, $\varphi \leq \psi$ iff $\varphi \subseteq \psi$, $\varphi\psi = \lambda s.\psi(\varphi(s))$, $(\varphi, \psi)(L(s)) = \varphi(s)$, $(\varphi, \psi)(R(s)) = \psi(s)$ and $(\varphi, \psi)(s)\uparrow$ otherwise. Prove that $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is an OS.

Hint. Follow the proof of 3.1.

Example 3.1 is a particular instance of example 4.7 with $M = \omega$.

Exercise 4.2 (Example 4.8). Let M, L, R, I be the same as in the previous exercise. Take $\mathscr{F} = \{\varphi/\varphi: M \to 2^M\}$, $\varphi \leq \psi$ iff $\forall s(\varphi(s) \subseteq \psi(s))$ and introduce $\circ, \Pi$ as above. Show that $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is an OS and the OS of example 4.7 is a subspace of it.

Here $\mathscr{F}$ consists of unary partial multiple-valued functions which can also be regarded as binary relations, i.e. $\mathscr{F} = \{\varphi/\varphi \subseteq M^2\}$.

The following exercise establishes the independence of the axioms A_1, A_2, A_3.

Exercise 4.3. Construct a 5-tuple $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ to meet all the OS-axioms but A_1 (respectively A_2, A_3).

Hint. Let J, L, R be as in example 4.5 and $J(0, 0) = 0$. (The first function J suggested there would do.) Take $\mathscr{F} = \{\varphi/\varphi: \omega \to \omega \,\&\, \varphi(0) = 0\}$, $I = \lambda s.s$,

$\varphi \le \psi$ iff $\forall s(\varphi(s) = 0 \vee \varphi(s) = \psi(s))$, $\varphi\psi = \lambda s.\varphi(\psi(s))$ and $(\varphi, \psi) = \lambda s.J(\varphi(s), \psi(s))$, then take $\varphi = \lambda s.0$ and show that $\varphi \le I$ but $(\varphi, I) \not\le (I, I)$. In the case of A_2 take an OS and replace Π, L, R by $\Pi_1 = \lambda\varphi\psi.(\varphi, I, \psi)$, $L_1 = L$ and $R_1 = R^2$. In the case of A_3 take $\Pi_1 = \lambda\varphi\psi.\varphi$.

Exercise 4.4. Show that the axiom A_1 can be replaced by its particular instance $\varphi \le \varphi_1 \Rightarrow (I, \varphi) \le (I, \varphi_1)$.

Hint. Make use of the equality

$$(\varphi, \psi) = (RL, R)(I, \psi L)(I, \varphi).$$

Exercise 4.5. Let $\mathscr{S}$ be an OS and $\mathscr{S}_1$ be obtained from it by introducing a new partial order, $\varphi \le_1 \psi$ iff $\varphi = \psi$. Show that $\mathscr{S}_1$ is also an OS.

It may well happen that a set $\mathscr{F}$ admits different operations $\circ, \Pi$ satisfying the OS-axioms, cf. examples 4.2, 4.5. If $\circ$ is fixed however, then the following exercise shows that Π is in a sense unique.

Exercise 4.6. Let $\mathscr{F}$ be a semigroup and both Π, L, R and Π_1, L_1, R_1 meet the axioms A_2, A_3. Prove that there are elements ρ, ρ_1 such that $(\varphi, \psi)_1 = \rho(\varphi, \psi)$ and $(\varphi, \psi) = \rho_1(\varphi, \psi)_1$ for all φ, ψ.

Hint. Take $\rho = (L, R)_1$, $\rho_1 = (L_1, R_1)$.

It follows by exercise 4.6 that all operations Π to meet A_2, A_3 are simultaneously monotonic or nonmonotonic, hence the semigroup suggested in the hint to exercise 4.3 can not be augmented with a pairing operation to become an OS.

Assume from now on that an OS $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is given.

Exercise 4.7. Show that the operations $\circ, \Pi$ are non-commutative, Π is non-associative and there is an element φ such that $\psi\varphi = I$ for no ψ.

Hint. $LR \ne RL$, $(I, R, I) \ne ((I, R), I)$ and $\psi L \ne I$ for all ψ.

Exercise 4.8. Prove the following equalities.

a. $RLR(R^2, (LR, RL)^4 L(I, I)^5, L)((I, R), L) = L.$
b. $R(\varphi R(\psi R, L)^3, L)^3(\varphi, \psi) = (\varphi\psi)^2\varphi.$
c. $L(\varphi LR, (\psi R, \psi L)(L, R^2))^{3n} = (\varphi\psi^2)^n L.$
d. $RL((L\chi, R)(\psi R, \varphi(R, L)), \varphi L)(\psi, R)(\chi, R) = \varphi R(L^2, R^2, \psi(\chi L, R^2))(I, I).$

Exercise 4.9. Let $K_0 = (R, L), K_1 = (I, I)$, $\langle\!\langle\ \rangle\!\rangle = \lambda\theta.(L, \theta R)$ and $\mathscr{B}_0 = \{R, K_1, \langle\!\langle K_0\rangle\!\rangle, \langle\!\langle K_1\rangle\!\rangle, \langle\!\langle\langle\!\langle L\rangle\!\rangle\rangle\!\rangle, \langle\!\langle\langle\!\langle R\rangle\!\rangle\rangle\!\rangle\}$. Prove that $cl(\{L, R\} \cup \mathscr{B}/\circ, \Pi) = cl(\mathscr{B}_0 \cup \langle\!\langle\mathscr{B}\rangle\!\rangle/\circ)$. (Of course, $\langle\!\langle\mathscr{B}\rangle\!\rangle$ stands for $\{\langle\!\langle\psi\rangle\!\rangle/\psi \in \mathscr{B}\}$.)

Hint: Use the equalities $\varphi = R\langle\!\langle\varphi\rangle\!\rangle K_1$, $\langle\!\langle\varphi\psi\rangle\!\rangle = \langle\!\langle\varphi\rangle\!\rangle\langle\!\langle\psi\rangle\!\rangle$, $(\varphi, \psi) = \langle\!\langle\psi\rangle\!\rangle K_0 \langle\!\langle\varphi\rangle\!\rangle K_1$ and $\langle\!\langle\langle\!\langle\varphi\rangle\!\rangle\rangle\!\rangle = K_3 \langle\!\langle\varphi\rangle\!\rangle K_4$, where $K_3 = \langle\!\langle\langle\!\langle R\rangle\!\rangle K_0 \langle\!\langle RL\rangle\!\rangle \circ K_1\rangle\!\rangle K_0 \langle\!\langle L^2\rangle\!\rangle K_1$, $K_4 = \langle\!\langle R^2\rangle\!\rangle K_0 \langle\!\langle\langle\!\langle L\rangle\!\rangle\rangle\!\rangle K_1$.

It is possible to consider an algebraic system alternative to OS, replacing Π by $\langle\!\langle\ \rangle\!\rangle$ and introducing suitable axioms.

CHAPTER 5

Iterative operative spaces

The aim of this chapter is to formally introduce the algebraic system of iterative operative space as the natural framework for the development of a general recursion theory.

Let $\mathscr{S}=(\mathscr{F},I,\Pi,L,R)$ be an OS. Consider the following axiom.

Axiom μA_0. Two additional unary operations $\langle\ \ \rangle, [\ \ ]$ called *translation* and *iteration* are given such that

$$
\begin{aligned}
&(£) && (\varphi L,\langle\varphi\rangle R)\le\langle\varphi\rangle,\\
& && R\psi\le\psi\psi_1\,\&(\varphi L\psi,\tau\psi_1)\le\tau\Rightarrow\langle\varphi\rangle\psi\le\tau.\\
&(££) && (I,\varphi[\varphi])\le[\varphi],\\
& && (\psi,\varphi\tau)\le\tau\Rightarrow[\varphi]\psi\le\tau.
\end{aligned}
$$

Taking $\psi=I,\psi_1=R$, one gets immediately that $\langle\varphi\rangle=\mu\theta.(\varphi L,\theta R)$ and $[\varphi]=\mu\theta.(I,\varphi\theta)$.

If the axiom μA_0 is satisfied, then $\mathscr{S}$ is said to be an *iterative operative space* (IOS). The operations $\langle\ \ \rangle, [\ \ ]$ are not included in the signature of IOS because their semantics is unambiguously implied by that of $\le,\circ$, Π, L, R.

For instance, the OS of examples 4.1, 4.6 are iterative respectively by 3.1, 3.3. The OS of example 4.3 is also iterative. It will be shown in the exercises that the OS of examples 4.7, 4.8 are iterative, while those of examples 4.2, 4.4, 4.5 are not.

Instead of the axiom μA_0, we shall use sometimes the stronger axioms $\mu A_1,\mu A_2$ or μA_3. Some auxiliary notions are needed to formulate them. To begin with, we introduce the notion of *inductive mapping* inductively as follows.

1. The mappings $\Gamma=\lambda\theta_1\ldots\theta_n.\theta_i$, $1\le i\le n$, and $\Gamma=\lambda\theta_1\ldots\theta_n.\psi$, $\psi\in\{I,L,R\}$, are inductive.

2. If $\Gamma_1,\Gamma_2:\mathscr{F}^n\to\mathscr{F}$ are inductive, then so are $\Gamma=\lambda\theta_1\ldots\theta_n.\Gamma_1(\theta_1,\ldots,\theta_n)\circ\Gamma_2(\theta_1,\ldots,\theta_n)$ and $\Gamma=\lambda\theta_1\ldots\theta_n.(\Gamma_1(\theta_1,\ldots,\theta_n),\Gamma_2(\theta_1,\ldots,\theta_n))$.

3. If $\Gamma_1:\mathscr{F}_1^{n+1}\to\mathscr{F}$ is inductive and for all $\theta_1,\ldots,\theta_n$ the element $\mu\theta.\Gamma_1(\theta_1,\ldots,\theta_n,\theta)$ exists, then $\Gamma=\lambda\theta_1\ldots\theta_n.\mu\theta.\Gamma_1(\theta_1,\ldots,\theta_n,\theta)$ is inductive.

A *regular segment* is a subset of $\mathscr{F}$ of the form $\{\theta/\alpha\theta\chi\le\tau$ for all $\langle\alpha,\chi,\tau\rangle\in\mathscr{A}\}$, where $\mathscr{A}\subseteq\mathscr{D}\times\mathscr{F}^2$. A *normal segment* is a set of the form $\{\theta/\theta\chi\le\tau$ for all $\langle\chi,\tau\rangle\in\mathscr{A}\}$, where $\mathscr{A}\subseteq\mathscr{F}^2$. In order to state the axiom μA_1 assume that the element $\langle\varphi\rangle=\mu\theta.(\varphi L,\theta R)$ exists for all φ and call *simple segments* the sets of the form $\{\theta/\theta\chi\le\tau\}$ or $\{\theta/\langle\theta\rangle\le\langle I\rangle\tau\}$.

Axiom μA_1 *($\mu A_2,\mu A_3$).* For any $n+1$-ary inductive mapping Γ and any

$\theta_1,\dots,\theta_n$ the inequality $\Gamma(\theta_1,\dots,\theta_n,\theta)\leq\theta$ has a solution which is a member of all simple (respectively normal, regular) segments closed under the mapping $\lambda\theta.\Gamma(\theta_1,\dots,\theta_n,\theta)$.

All normal segments are regular since $I\in\mathscr{D}$; thus axiom μA_3 implies axiom μA_2. Further connections between the four μ-axioms are established below. In each particular instance the μ-axiom used will be indicated by the corresponding number of asterisks, e.g. the proof of 5.7** makes use of μA_2.

Proposition 5.1. All inductive mappings are monotonic.

Proof. The mappings $\Gamma=\lambda\theta_1\dots\theta_n.\theta_i$ and $\Gamma=\lambda\theta_1\dots\theta_n.\psi$ are monotonic.

If $\Gamma_1,\Gamma_2:\mathscr{F}^n\to\mathscr{F}$ are monotonic, then so are

$$\Gamma=\lambda\theta_1\dots\theta_n.\Gamma_1(\theta_1,\dots,\theta_n)\Gamma_2(\theta_1,\dots,\theta_n)$$

and

$$\Gamma=\lambda\theta_1\dots\theta_n.(\Gamma_1(\theta_1,\dots,\theta_n),\Gamma_2(\theta_1,\dots,\theta_n))$$

since the monotonicity of $\circ,\Pi$.

Let $\Gamma_1:\mathscr{F}^{n+1}\to\mathscr{F}$ be monotonic and

$$\Gamma=\lambda\theta_1\dots\theta_n.\mu\theta.\Gamma_1(\theta_1,\dots,\theta_n,\theta).$$

If $\theta_1\leq\tau_1,\dots,\theta_n\leq\tau_n$, then

$$\Gamma_1(\theta_1,\dots,\theta_n,\Gamma(\tau_1,\dots,\tau_n))\leq\Gamma_1(\tau_1,\dots,\tau_n,\Gamma(\tau_1,\dots,\tau_n))\leq\Gamma(\tau_1,\dots,\tau_n),$$

hence $\Gamma(\theta_1,\dots,\theta_n)\leq\Gamma(\tau_1,\dots,\tau_n)$ since $\Gamma(\theta_1,\dots,\theta_n)=\mu\theta.\Gamma_1(\theta_1,\dots,\theta_n,\theta)$. Therefore, Γ is monotonic. The proof is complete.

Proposition 5.2* (5.2).** Let θ_0 be the element assumed in the axiom μA_1 (respectively $\mu A_2,\mu A_3$). Then $\theta_0=\mu\theta.\Gamma(\theta_1,\dots,\theta_n,\theta)$.

Proof. Since $\Gamma(\theta_1,\dots,\theta_n,\theta_0)\leq\theta_0$, it suffices to show that whenever $\Gamma(\theta_1,\dots,\theta_n,\tau)\leq\tau$, then $\theta_0\leq\tau$.

Suppose that $\Gamma(\theta_1,\dots,\theta_n,\tau)\leq\tau$ and consider the simple (respectively normal) segment $\mathscr{E}=\{\theta/\theta\leq\tau\}$. Whenever $\theta\in\mathscr{E}$, then

$$\Gamma(\theta_1,\dots,\theta_n,\theta)\leq\Gamma(\theta_1,\dots,\theta_n,\tau)\leq\tau$$

by the monotonicity of Γ, hence $\mathscr{E}$ is closed under the mapping $\lambda\theta.\Gamma(\theta_1,\dots,\theta_n,\theta)$, which implies $\theta_0\in\mathscr{E}$ by μA_1 (by μA_2). Thereby the proof is completed.

Notice that the element $\mu\theta.\Gamma(\theta_1,\dots,\theta_n,\theta)$ above is also a least fixed point of $\lambda\theta.\Gamma(\theta_1,\dots,\theta_n,\theta)$ since the monotonicity of this mapping.

It follows from this last statement that each of the axioms $\mu A_1,\mu A_2,\mu A_3$ ensures the existence of the elements $\langle\varphi\rangle=\mu\theta.(\varphi L,\theta R)$ and $[\varphi]=\mu\theta.(I,\varphi\theta)$ for all φ.

The following two statements show that the axiom μA_1 implies both (£) and (££), i.e. μA_1 implies μA_0.

Proposition 5.3*. If $R\psi\leq\psi\psi_1$ and $(\varphi L\psi,\tau\psi_1)\leq\tau$, then $\langle\varphi\rangle\psi\leq\tau$.

Proof. The set $\mathscr{E}=\{\theta/\theta\psi\leq\tau\}$ is a simple segment. If $\theta\in\mathscr{E}$, then

$$(\varphi L,\theta R)\psi=(\varphi L\psi,\theta R\psi)\leq(\varphi L\psi,\theta\psi\psi_1)\leq(\varphi L\psi,\tau\psi_1)\leq\tau;$$

hence $\langle\varphi\rangle\in\mathscr{E}$ by μA_1. Thus the proof is complete.

Proposition 5.4*. If $(\psi, \varphi\tau) \leq \tau$, then $[\varphi]\psi \leq \tau$.

Proof. The set $\mathscr{E} = \{\theta/\theta\psi \leq \tau\}$ is a simple segment and if $\theta \in \mathscr{E}$, then

$$(I, \varphi\theta)\psi = (\psi, \varphi\theta\psi) \leq (\psi, \varphi\tau) \leq \tau.$$

Therefore, $[\varphi] \in \mathscr{E}$ by μA_1, which completes the proof.

Our next aim is to show that μA_2 implies μA_1.

Proposition 5.5. $L\langle\varphi\rangle = \varphi L$ and $R\langle\varphi\rangle = \langle\varphi\rangle R$.

This follows from the equality $\langle\varphi\rangle = (\varphi L, \langle\varphi\rangle R)$.

Proposition 5.6. $\bar{n}\langle\varphi\rangle = \varphi\bar{n}$ for all n.

Proof.

$$\bar{n}\langle\varphi\rangle = LR^n\langle\varphi\rangle = L\langle\varphi\rangle R^n = \varphi LR^n = \varphi\bar{n}.$$

Proposition 5.7.** If $\forall n(\varphi\bar{n}\psi \leq \rho\bar{n}\sigma)$, then $\langle\varphi\rangle\psi \leq \langle\rho\rangle\sigma$.

Proof. Consider the normal segment

$$\mathscr{E} = \{\theta/\forall n(\theta R^n\psi \leq \langle\rho\rangle R^n\sigma)\}.$$

If $\theta \in \mathscr{E}$, then

$$\begin{aligned}(\varphi L, \theta R)R^n\psi &= (\varphi\bar{n}\psi, \theta R^{n+1}\psi) \leq (\rho\bar{n}\sigma, \langle\rho\rangle R^{n+1}\sigma)\\ &= (\rho L, \langle\rho\rangle R)R^n\sigma = \langle\rho\rangle R^n\sigma,\end{aligned}$$

hence $\langle\varphi\rangle \in \mathscr{E}$ by μA_2. Taking $n = 0$, one gets $\langle\varphi\rangle\psi \leq \langle\rho\rangle\sigma$.

Proposition 5.8.** $\langle\varphi\rangle\psi \leq \langle\rho\rangle\sigma$ iff $\forall n(\varphi\bar{n}\psi \leq \rho\bar{n}\sigma))$.

This follows from 5.6, 5.7**.

Proposition 5.9.** $\langle\varphi\rangle\psi = \langle\rho\rangle\sigma$ iff $\forall n(\varphi\bar{n}\psi = \rho\bar{n}\sigma)$.

This follows from 5.8**.

Proposition 5.10.** All simple segments are normal.

Proof. If $\mathscr{E} = \{\theta/\theta\chi \leq \tau\}$, then $\mathscr{E}$ is a normal segment. If $\mathscr{E} = \{\theta/\langle\theta\rangle \leq \langle I\rangle\tau\}$, then $\mathscr{E} = \{\theta/\forall n(\theta\bar{n} \leq \bar{n}\tau)\}$ by 5.8**, hence $\mathscr{E}$ is again a normal segment. The proof is complete.

As an immediate corollary to 5.10** one obtains that μA_2 implies μA_1, thus

$$\mu A_3 \Rightarrow \mu A_2 \Rightarrow \mu A_1 \Rightarrow \mu A_0.$$

The sufficient condition $(*)_0$ given in chapter 18 verifies μA_2 but not μA_3, hence $\mu A_2 \not\Rightarrow \mu A_3$. I conjecture that $\mu A_1 \not\Rightarrow \mu A_2$ and $\mu A_0 \not\Rightarrow \mu A_1$.

The proof of the Normal Form Theorem and the First Recursion Theorem given in chapter 9 imply that it is sufficient to allow in $\mu A_1, \mu A_2, \mu A_3$ the mappings $\lambda\theta_1\theta.(\theta_1 L, \theta R)$, $\lambda\theta_1\theta.(I, \theta_1\theta)$ and $\lambda\theta_1\theta.\bar{1}[\theta_1(I, \langle\theta\rangle)]$ only. Therefore, axiom scheme μA_1 is equivalent to a single first order axiom. The axioms $\mu A_2, \mu A_3$ are second order ones, though the normal and regular segments we consider will often be introduced by means of countable and effectively given sets $\mathscr{A}$.

It almost always suffices to allow in $\mu A_2, \mu A_3$ just the mappings $\lambda\theta_1\theta.(\theta_1 L, \theta R)$ and $\lambda\theta_1\theta.(I, \theta_1\theta)$; axiom μA_3 restricted this way is exactly the basic μ-axiom in Ivanov [1980]. The first version of a μ-axiom for OS was μA_3, while $\mu A_0, \mu A_1$ were invented later. (The point (££) was borrowed from Skordev [1982].)

The requirements of (£) can be restricted to the element $\langle I \rangle$, provided the equality $(\varphi L, \theta R) = \theta$ has a solution for all φ.

Proposition 5.11. Let $\langle I \rangle$ be an element satisfying (£), i.e. $(L, \langle I \rangle R) \leq \langle I \rangle$ and whenever $R\psi \leq \psi\psi_1$, $(L\psi, \tau\psi_1) \leq \tau$, then $\langle I \rangle\psi \leq \tau$. Let $\langle\quad\rangle_1$ be a unary operation over $\mathscr{F}$ such that $(\varphi L, \langle\varphi\rangle_1 R) = \langle\varphi\rangle_1$ for all φ. Then the operation $\langle\quad\rangle = \lambda\varphi.\langle I \rangle\langle\varphi\rangle_1$ satisfies (£).

Proof. It follows that

$$(\varphi L, \langle\varphi\rangle R) = (\varphi L, \langle I \rangle\langle\varphi\rangle_1 R) = (L, \langle I \rangle R)(\varphi L, \langle\varphi\rangle_1 R) = \langle I \rangle\langle\varphi\rangle_1 = \langle\varphi\rangle.$$

If $R\psi \leq \psi\psi_1$ and $(\varphi L\psi, \tau\psi_1) \leq \tau$, then $R\langle\varphi\rangle_1\psi \leq \langle\varphi\rangle_1\psi\psi_1$ and $(L\langle\varphi\rangle_1\psi, \tau\psi_1) \leq \tau$; hence $\langle I \rangle\langle\varphi\rangle_1\psi \leq \tau$, i.e. $\langle\varphi\rangle\psi \leq \tau$. This completes the proof.

Proposition 5.12. $L[\varphi] = I$, $R[\varphi] = \varphi[\varphi]$.

This follows from the equality $[\varphi] = (I, \varphi[\varphi])$.

Proposition 5.13. Suppose that axiom μA_3 is satisfied at least by the mapping $\lambda\theta_1\theta.(I, \theta_1\theta)$ and let $\langle\quad\rangle_1$ be a unary operation such that $\bar{n}\langle\varphi\rangle_1 = \varphi\bar{n}$ for all n. (In particular, the last equality will hold whenever $(\varphi L, \langle\varphi\rangle_1 R) = \langle\varphi\rangle_1$.) Then axiom μA_0 and the assertion of 5.7** hold.

Proof. Take $\sigma = \langle(\varphi L^2, LR^2)\rangle_1[\langle(RL, R^2)\rangle_1]$ and $\langle\varphi\rangle = \bar{1}[\sigma]$ by definition. Notice that $\bar{n}\sigma = (\varphi\bar{n}L, \overline{n+2})$ for all n.

Consider the regular segment

$$\mathscr{E} = \{\theta / L\theta \leq I \,\&\, \forall n(\overline{n+1}\theta R \leq \overline{n+2}[\sigma])\}.$$

If $\theta \in \mathscr{E}$, then $L(I, \sigma\theta) = I$ and

$$\begin{aligned}\overline{n+1}(I, \sigma\theta)R &= \bar{n}\sigma\theta R = (\varphi\bar{n}L, \overline{n+2})\theta R = (\varphi\bar{n}L\theta R, \overline{n+2}\theta R)\\ &\leq (\varphi\overline{n+1}, \overline{n+3}[\sigma]) = (\varphi\overline{n+1}L, \overline{n+3})[\sigma]\\ &= \overline{n+1}\sigma[\sigma] = \overline{n+2}[\sigma];\end{aligned}$$

hence $[\sigma] \in \mathscr{E}$. We get in particular $\bar{1}[\sigma]R \leq \bar{2}[\sigma]$; hence

$$(\varphi L, \langle\varphi\rangle R) = (\varphi L, \bar{1}[\sigma]R) \leq (\varphi L, \bar{2}[\sigma]) = (\varphi L^2, \bar{2})[\sigma] = \bar{0}\sigma[\sigma] = \bar{1}[\sigma] = \langle\varphi\rangle.$$

Let $R\psi \leq \psi\psi_1$ and $(\varphi L\psi, \tau\psi_1) \leq \tau$. Consider the regular segment

$$\mathscr{E} = \{\theta / L\theta \leq I \,\&\, \forall n(\overline{n+1}\theta\psi \leq \tau\psi_1^n)\}.$$

If $\theta \in \mathscr{E}$, then

$$\begin{aligned}\overline{n+1}(I, \sigma\theta)\psi &= \bar{n}\sigma\theta\psi = (\varphi\bar{n}L, \overline{n+2})\theta\psi = (\varphi\bar{n}L\theta\psi, \overline{n+2}\theta\psi)\\ &\leq (\varphi\bar{n}\psi, \tau\psi_1^{n+1}) \leq (\varphi L\psi\psi_1^n, \tau\psi_1^{n+1}) = (\varphi L\psi, \tau\psi_1)\psi_1^n \leq \tau\psi_1^n,\end{aligned}$$

hence $[\sigma]\in\mathscr{E}$. Taking $n=0$, we get $1[\sigma]\psi\leq\tau$, i.e. $\langle\varphi\rangle\psi\leq\tau$. Therefore, (£) is valid, while (££) follows by the proof of 5.4*.

Suppose that $\varphi\bar{n}\psi\leq\rho\bar{n}\tau$ for all n. In order to show that $\langle\varphi\rangle\psi\leq\langle\rho\rangle\tau$ take the regular segment

$$\mathscr{E}=\{\theta/L\theta\leq I\,\&\,\forall n(\overline{n+1}\theta\psi\leq\langle\rho\rangle R^n\tau)\}.$$

If $\theta\in\mathscr{E}$, then

$$\begin{aligned}\overline{n+1}(I,\sigma\theta)\psi&=(\varphi\bar{n}L,\overline{n+2})\theta\psi\leq(\varphi\bar{n}\psi,\langle\rho\rangle R^{n+1}\tau)\leq(\rho\bar{n}\tau,\langle\rho\rangle R^{n+1}\tau)\\&=(\rho L,\langle\rho\rangle R)R^n\tau=\langle\rho\rangle R^n\tau,\end{aligned}$$

hence $[\sigma]\in\mathscr{E}$. Taking $n=0$, we get the desired inequality, which completes the proof.

It may happen that an OS is iterative but a subspace of it is not, e.g. the OS of example 4.1 and its subspace of example 4.2. However, the following statement shows that the axiom μA_0 is in a sense hereditary.

Proposition 5.14. Let $\mathscr{S}$ be an OS, let $\mathscr{S}_1$ be a subspace of it. Suppose that $\mathscr{S}$ satisfies μA_0 and the semigroup of $\mathscr{S}$ is closed under $\langle\ \ \rangle$ and $[\ \ ]$. Then $\mathscr{S}_1$ satisfies μA_0 and is a subspace of $\mathscr{S}$ as an IOS.

The proof is trivial. Both (£) and (££) are hereditary.

EXERCISES TO CHAPTER 5

Exercise 5.1. Prove that the OS of example 4.7 is iterative. Writing L_1, R_1 respectively for L^{-1}, R^{-1}, show that

$$\langle\varphi\rangle=\bigcup_n R_1^n L_1\varphi L R^n \text{ and } [\varphi]=\bigcup_n (R_1\varphi)^n L_1.$$

Hint. Follow the proofs of 3.1, 3.2.

Exercise 5.2. Prove that the OS $\mathscr{S}$ of example 4.8 is iterative and $\langle\ \ \rangle, [\ \ ]$ can be characterized as in the previous exercise. Show that the IOS of example 4.7 is a subspace of $\mathscr{S}$.

Hint. Follow the proofs of 3.1, 3.2 again.

Exercise 5.3. Let $\mathscr{S}$ be the IOS of example 4.7 or example 4.8. Prove that whenever $\Gamma:\mathscr{F}\to\mathscr{F}$ is monotonic, then it has a least fixed point which is a member of all regular segments $\mathscr{E}$ closed under Γ. Therefore, $\mathscr{S}$ is μA_3-iterative.

Hint. Take $\theta_\xi=\sup\{\Gamma(\theta_\eta)/\eta<\xi\}$ for all ξ. The increasing transfinite sequence $\{\theta_\xi\}$ can not consist of distinct members, hence there is a ζ such that $\theta_\zeta=\theta_{\zeta+1}$. Therefore, θ_ζ is a fixed point of Γ. Using the fact that $\mathscr{E}$ is closed under least upper bounds, show by transfinite induction that $\theta_\xi\in\mathscr{E}$ for all ξ, hence $\theta_\zeta\in\mathscr{E}$. (It suffices to consider the sequence $\{\theta_\xi\}_{\xi\leq\omega}$, provided Γ is continuous with respect to least upper bounds of increasing countable sequences. And the inductive mappings are continuous in this particular instance.)

Exercise 5.4. Prove that the IOS of example 3.2 has the property considered in the previous exercise.

Exercise 5.5. Prove that if (££) holds, then the axiom $\psi \leq \psi_1 \Rightarrow \varphi\psi \leq \varphi\psi_1$ can be replaced by the weaker one $\psi \leq \psi_1 \Rightarrow L\psi \leq L\psi_1$, $R\psi \leq R\psi_1$.

Hint. Using (££), show that $[\varphi L] \leq (I, \varphi)$. The inequality $(I, \varphi L[\varphi L]) \leq [\varphi L]$ implies both $I \leq L[\varphi L]$ and $\varphi L[\varphi L] \leq R[\varphi L]$, hence $I = L[\varphi L]$ and $\varphi = \varphi L[\varphi L] \leq R[\varphi L]$. Therefore $\varphi = R[\varphi L]$. Show that $\psi \leq \psi_1$ implies $[\varphi L]\psi \leq [\varphi L]\psi_1$ by (££); multiplying on the left by R, obtain $\varphi\psi \leq \varphi\psi_1$.

Exercise 5.6. Let $\mathscr{S}$ be an OS with $\varphi \leq \psi$ iff $\varphi = \psi$. Show that $\mathscr{S}$ does not satisfy (££).

Hint. Suppose that the equality $(I, R\theta) = \theta$ has a least solution denoted by $[R]$. However, 'least' means 'unique' in this space, hence $(I, R(I, \varphi)) = (I, \varphi)$ implies $(I, \varphi) = [R]$ for all φ, which is not the case.

It follows from this last exercise that the OS of examples 4.2, 4.4, 4.5 are not iterative.

CHAPTER 6

Translation and iteration lemmas

In this chapter we study the operations of translation and iteration. A new operation called primitive recursion is also introduced. The independence of the IOS-operations is examined in the exercises.

Assume that an IOS $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is given. Recall that $\langle\ \ \rangle = \lambda\varphi.\mu\theta.(\varphi L, \theta R)$ and $[\ \ ] = \lambda\varphi.\mu\theta.(I, \varphi\theta)$, in particular, $(\varphi L, \langle\varphi\rangle R) = \langle\varphi\rangle$ and $(I, \varphi[\varphi]) = [\varphi]$ for all φ.

Proposition 6.1. Iteration is monotonic.
Follows by 5.1.

Proposition 6.2. $[L] = (I, I)$.
Proof. Using 5.12, we get

$$[L] = (I, L[L]) = (I, I).$$

Proposition 6.3. $\bar{n}[I] = I$ for all n.
Proof. We have $\bar{0}[I] = L[I] = I$ and

$$\overline{n+1}\,[I] = \bar{n}R[I] = \bar{n}I[I] = \bar{n}[I],$$

which completes the proof.

The partially ordered set $\mathscr{F}$ has a least member, namely $O = R[R]$.

Proposition 6.4. $O \leq \varphi$ for all φ.
Proof. The equality $(I, R(I, \varphi)) = (I, \varphi)$ implies $[R] \leq (I, \varphi)$ by (££). Therefore, $O = R[R] \leq \varphi$.

Proposition 6.5. $LO = RO = O$.
Proof. Using 6.4, we get

$$LO \leq L(O, O) = O,$$

hence $LO = O$. Similarly, $RO = O$. This direct proof is due to N. Georgieva.

As an immediate corollary to 6.5 $\alpha O = O$ for all $\alpha \in \mathscr{D}$. However, the statement $\forall\varphi(\varphi O = O)$ is not a theorem; it is valid in examples 4.1, 4.3, 4.7, 4.8 but fails in example 4.6.

Proposition 6.6. $O\varphi = O$ for all φ.

Proof. It suffices to show that $O\varphi \le O$. The equality $(\varphi, R(\varphi, O)) = (\varphi, O)$ implies $[R]\varphi \le (\varphi, O)$ by (££), hence $O\varphi \le O$.

Proposition 6.7. $[\varphi] = (I, R[\varphi])$.

This follows from 5.12.

Proposition 6.8. $\varphi = R[\varphi L]$.

Proof.

$$R[\varphi L] = \varphi L[\varphi L] = \varphi I = \varphi.$$

Proposition 6.9. $(\varphi, \psi) = A[\varphi L^2][\psi L]$, where $A = (R, RL)$.

Proof.

$$A[\varphi L^2][\psi L] = (\varphi L^2[\varphi L^2], R)[\psi L] = (\varphi L[\psi L], R[\psi L]) = (\varphi, \psi).$$

The above equality is due to N. Georgieva and corresponds to a similar one of Skordev [1982].

Proposition 6.10. $\chi[\varphi\chi]\psi = \mu\theta.\chi(\psi, \varphi\theta)$. In particular, $[\varphi]\psi = \mu\theta.(\psi, \varphi\theta)$.

Proof. We have

$$\chi(\psi, \varphi\chi[\varphi\chi]\psi) = \chi(I, \varphi\chi[\varphi\chi])\psi = \chi[\varphi\chi]\psi.$$

Whenever $\chi(\psi, \varphi\tau) \le \tau$, then $(\psi, \varphi\chi(\psi, \varphi\tau)) \le (\psi, \varphi\tau)$; hence $[\varphi\chi]\psi \le (\psi, \varphi\tau)$ by (££). Therefore, $\chi[\varphi\chi]\psi \le \chi(\psi, \varphi\tau) \le \tau$, which completes the proof.

Proposition 6.11. $R[\varphi]\psi = \mu\theta.\varphi(\psi, \theta)$.

This follows from 5.12, 6.10.

The operation $[\ \ ]_1 = \lambda\varphi.\mu\theta.\varphi(I, \theta)$ is a natural alternative to iteration, replacing (££) by $(££)_1$:

$$\varphi(I, [\varphi]_1) \le [\varphi]_1,$$
$$\varphi(\psi, \tau) \le \tau \Rightarrow [\varphi]_1\psi \le \tau.$$

Proposition 6.11 gives $[\varphi]_1 = R[\varphi]$; hence $[\varphi] = (I, [\varphi]_1)$.

Proposition 6.12. $R[\varphi(\psi L, \chi R)]\rho = \mu\theta.\varphi(\psi\rho, \chi\theta)$.

This follows from 6.11 and the equality

$$\varphi(\psi\rho, \chi\theta) = \varphi(\psi L, \chi R)(\rho, \theta).$$

Proposition 6.13. $[\varphi]\psi = R[\sigma]$, where $\sigma = (\psi L, \varphi R)$.

Proof.

$$[\varphi]\psi = \mu\theta.(\psi, \varphi\theta) = R[(\psi L, \varphi R)]$$

by 6.10, 6.12.

Proposition 6.14. $\varphi[\psi] = \bar{1}[\sigma]$, where $\sigma = (\varphi R^2, \bar{0}, \psi R^2)$.

Proof. We have

$$(I, \psi R^2[\sigma]) = (\bar{0}, \psi R^2)[\sigma] = R\sigma[\sigma] = R^2[\sigma];$$

hence $[\psi] \leq R^2[\sigma]$. Multiplying on the left by φ, we get

$$\varphi[\psi] \leq \varphi R^2[\sigma] = L\sigma[\sigma] = \bar{1}[\sigma].$$

Let $\tau = (I, \varphi[\psi], [\psi])$. Then

$$(I, \sigma\tau) = (I, \varphi[\psi], I, \psi[\psi]) = \tau,$$

hence $[\sigma] \leq \tau$. Therefore, $\bar{1}[\sigma] \leq \bar{1}\tau = \varphi[\psi]$. The proof is complete.

Proposition 6.15. $[\varphi]\psi[\chi] = \bar{1}[\sigma]$, where $\sigma = ((\psi R^2, \varphi\bar{1}), \bar{0}, \chi R^2)$.

Proof. To begin with, $(I, \chi R^2[\sigma]) = (\bar{0}, \chi R^2)[\sigma] = R\sigma[\sigma] = R^2[\sigma]$ implies $[\chi] \leq R^2[\sigma]$. Using this inequality, we get

$$(\psi[\chi], \varphi\bar{1}[\sigma]) \leq (\psi R^2[\sigma], \varphi\bar{1}[\sigma]) = (\psi R^2, \varphi\bar{1})[\sigma] = L\sigma[\sigma] = \bar{1}[\sigma],$$

hence $[\varphi]\psi[\chi] \leq \bar{1}[\sigma]$ by (££).

On the other hand, writing τ for $(I, [\varphi]\psi[\chi], [\chi])$ we get

$$(I, \sigma\tau) = (I, (\psi[\chi], \varphi[\varphi]\psi[\chi]), I, \chi[\chi]) = (I, (I, \varphi[\varphi])\psi[\chi], [\chi]) = \tau;$$

hence $[\sigma] \leq \tau$. Therefore, $\bar{1}[\sigma] \leq [\varphi]\psi[\chi]$, which completes the proof.

Proposition 6.16. $[\varphi[\psi]] = \bar{1}[\sigma]$, where $\sigma = ((\bar{0}, \varphi R^2), \bar{1}, \psi R^2)$.

Proof. We have $(\bar{1}[\sigma], \psi R^2[\sigma]) = (\bar{1}, \psi R^2)[\sigma] = R\sigma[\sigma] = R^2[\sigma]$; hence $[\psi]\bar{1}[\sigma] \leq R^2[\sigma]$. Using this inequality, we get

$$(I, \varphi[\psi]\bar{1}[\sigma]) \leq (I, \varphi R^2[\sigma]) = (\bar{0}, \varphi R^2)[\sigma] = L\sigma[\sigma] = \bar{1}[\sigma],$$

hence $[\varphi[\psi]] \leq \bar{1}[\sigma]$.

Let $\tau = (I, [\varphi[\psi]], [\psi][\varphi[\psi]])$. Then

$$\begin{aligned}(I, \sigma\tau) &= (I, (I, \varphi[\psi][\varphi[\psi]]), [\varphi[\psi]], \psi[\psi][\varphi[\psi]]) \\ &= (I, [\varphi[\psi]], (I, \psi[\psi])[\varphi[\psi]]) = \tau,\end{aligned}$$

hence $[\sigma] \leq \tau$. Therefore, $\bar{1}[\sigma] \leq [\varphi[\psi]]$. The proof is complete.

Proposition 6.17. Translation is monotonic.

This follows from 5.1.

Proposition 6.18. Translation is injective.

Proof. $\varphi = L\langle\varphi\rangle(I, I)$.

By contrast, iteration is not injective. For instance, $[R] = [R^2]$, while $R \neq R^2$ by 4.4. Actually, $[R] = (I, O) \leq [R^2]$. On the other hand,

$$(I, R^2[R]) = (I, RO) = (I, O) = [R]$$

implies $[R^2] \leq [R]$.

Proposition 6.19. If $\varphi L\psi \leq \rho L\tau$, $R\psi \leq \psi\psi_1$ and $\tau\psi_1 \leq R\tau$ for a certain ψ_1, then $\langle\varphi\rangle\psi \leq \langle\rho\rangle\tau$.

Proof. We have $R\psi \leq \psi\psi_1$ and

$$(\varphi L\psi, \langle \rho \rangle \tau\psi_1) \leq (\rho L\tau, \langle \rho \rangle R\tau) = (\rho L, \langle \rho \rangle R)\tau = \langle \rho \rangle \tau,$$

hence $\langle \varphi \rangle \psi \leq \langle \rho \rangle \tau$ by (£).

Proposition 6.20. If $\varphi L\psi = \rho L\tau$, $R\psi = \psi\psi_1$ and $\tau\psi_1 = R\tau$ for some ψ_1, then $\langle \varphi \rangle \psi = \langle \rho \rangle \tau$.

This follows from 6.19.

Proposition 6.21. $\langle \varphi \rangle \langle \psi \rangle = \langle \varphi\psi \rangle$.

Proof. Take $\psi_1 = R$, $\rho = \varphi\psi$ and $\tau = I$. Then the equalities $\varphi L\langle \psi \rangle = \varphi\psi L$, $R\langle \psi \rangle = \langle \psi \rangle R$ imply $\langle \varphi \rangle \langle \psi \rangle = \langle \varphi\psi \rangle$ by 6.20.

Proposition 6.22. If $(\varphi L, \tau R) = \tau$, then $\langle \varphi \rangle = \langle I \rangle \tau$.

This follows from 6.20.

We get in particular that $\langle \varphi \rangle$ is a unique solution to the equation $(\varphi L, \theta R) = \theta$, provided $\langle I \rangle = I$. (The last equality holds in examples 4.1, 4.6 but fails in example 4.3.) By contrast, there is no space in which the equation $(I, \varphi\theta) = \theta$ has a unique solution for each φ. Indeed, both $[L]$ and $[R]$ satisfy the equality $(I, R\theta) = \theta$ by 6.7 but are not equal.

Proposition 6.23. $R[\langle \varphi \rangle] = [\langle \varphi \rangle]\varphi$.

Proof. Using 6.13, we get

$$[\langle \varphi \rangle]\varphi = R[(\varphi L, \langle \varphi \rangle R)] = R[\langle \varphi \rangle].$$

Proposition 6.24. $\bar{n}[\langle \varphi \rangle] = \varphi^n$ for all n.

Proof.

$$\bar{n}[\langle \varphi \rangle] = LR^n[\langle \varphi \rangle] = L[\langle \varphi \rangle]\varphi^n = \varphi^n.$$

The binary operation $\Delta = \lambda\varphi\psi.\langle \varphi \rangle[\langle \psi \rangle]$ is called *primitive recursion.*

Proposition 6.25. $\Delta(\varphi, \psi) = \langle \varphi \rangle \Delta(I, \psi)$.

Proof.

$$\Delta(\varphi, \psi) = \langle \varphi \rangle[\langle \psi \rangle] = \langle \varphi \rangle \langle I \rangle [\langle \psi \rangle] = \langle \varphi \rangle \Delta(I, \psi).$$

Proposition 6.26. $\langle I \rangle \Delta(\varphi, \psi) = \Delta(\varphi, \psi)$.

This follows from the equality $\langle I \rangle \langle \varphi \rangle = \langle \varphi \rangle$.

Proposition 6.27. $L\Delta(\varphi, \psi) = \varphi$, $R\Delta(\varphi, \psi) = \Delta(\varphi, \psi)\psi$.

Proof.

$$L\Delta(\varphi, \psi) = \varphi L[\langle \psi \rangle] = \varphi, \quad R\Delta(\varphi, \psi) = \langle \varphi \rangle R[\langle \psi \rangle]$$
$$= \langle \varphi \rangle [\langle \psi \rangle]\psi = \Delta(\varphi, \psi)\psi.$$

Proposition 6.28. $\bar{n}\Delta(\varphi, \psi) = \varphi\psi^n$ for all n.

This follows from 6.27.

Proposition 6.29. $(\varphi,\psi) = B\Delta(L, B\Delta(\varphi, R))B\Delta(I,\psi)$, where $B = (LR, L)$.
Proof.

$$B\Delta(L, B\Delta(\varphi, R))B\Delta(I,\psi) = B\Delta(L,(\varphi R, \varphi))(\psi, I) = (\varphi R, L)(\psi, I) = (\varphi, \psi).$$

Proposition 6.30. If $\psi\chi \leq \chi\psi_1$ and $(\varphi\chi, \tau\psi_1) \leq \tau$ for some ψ_1, then $\Delta(\varphi,\psi)\chi \leq \tau$.
Proof. The inequalities

$$R[\langle\psi\rangle]\chi = [\langle\psi\rangle]\psi\chi \leq [\langle\psi\rangle]\chi\psi_1, \quad (\varphi L[\langle\psi\rangle]\chi, \tau\psi_1) = (\varphi\chi, \tau\psi_1) \leq \tau$$

imply $\langle\varphi\rangle[\langle\psi\rangle]\chi \leq \tau$ by (£).

Proposition 6.31. If $\varphi\chi = L\tau$, $\psi\chi = \chi\psi_1$ and $\tau\psi_1 = R\tau$ for some ψ_1, then $\Delta(\varphi,\psi)\chi = \langle I\rangle\tau$.
Proof. The equalities

$$\varphi L[\langle\psi\rangle]\chi = \varphi\chi = L\tau, \quad R[\langle\psi\rangle]\chi = [\langle\psi\rangle]\psi\chi = [\langle\psi\rangle]\chi\psi_1$$

and $\tau\psi_1 = R\tau$ imply $\langle\varphi\rangle[\langle\psi\rangle]\chi = \langle I\rangle\tau$ by 6.20.

One gets $\Delta(\varphi,\psi)\chi = \tau$, provided τ is of the form $\langle\tau_1\rangle\tau_2$. In particular, $\Delta(\varphi,\psi)\chi = \Delta(\varphi\chi,\psi_1)$ whenever $\psi\chi = \chi\psi_1$.

The following result shows that translation is a particular instance of primitive recursion.

Proposition 6.32. $\langle\varphi\rangle = \Delta(\varphi L, R)$.
This follows from 6.31. Take $\chi = I$, $\psi_1 = R$, $\tau = \langle\varphi\rangle$.

Proposition 6.33. $\Delta(\varphi,\psi) = \mu\theta.(\varphi, \theta\psi)$.
Proof. Using 6.25, 6.27, we get

$$\begin{aligned}(\varphi, \Delta(\varphi,\psi)\psi) = (\varphi, \langle\varphi\rangle\Delta(I,\psi)\psi) &= (\varphi L, \langle\varphi\rangle R)\Delta(I,\psi)\\ &= \langle\varphi\rangle\Delta(I,\psi) = \Delta(\varphi,\psi).\end{aligned}$$

If $(\varphi, \tau\psi) \leq \tau$, then $\Delta(\varphi,\psi) \leq \tau$ by 6.30.

Proposition 6.34. If $(\varphi, \tau\psi) = \tau$, then $\Delta(\varphi,\psi) = \langle I\rangle\tau$.
This follows from 6.31.

In particular, $\Delta(\varphi,\psi)$ is a unique solution to the equality $(\varphi, \theta\psi) = \theta$, provided $\langle I\rangle = I$.

Proposition 6.35. Let $C = \Delta((L^2, LR), \ (RL, R^2))$. Then $\bar{n}C = (\bar{n}L, \bar{n}R) = (\bar{n}L, \overline{n+1})$ for all n.
Proof.

$$\bar{n}C = (L^2, LR)(RL, R^2)^n = (L^2, LR)(R^nL, R^{n+1}) = (LR^nL, LR^{n+1}).$$

It is worth mentioning that $\langle I\rangle C = C$ by 6.26. Notice also that $C = \Delta(B^2, A^2)$.

Proposition 6.36. $\langle(\varphi,\psi)\rangle = C(\langle\varphi\rangle,\langle\psi\rangle)$.

Proof. The equalities

$$(L^2, LR)(\langle\varphi\rangle,\langle\psi\rangle) = (\varphi,\psi)L = L\langle(\varphi,\psi)\rangle,$$
$$(RL, R^2)(\langle\varphi\rangle,\langle\psi\rangle) = (\langle\varphi\rangle,\langle\psi\rangle)R,$$
$$\langle(\varphi,\psi)\rangle R = R\langle(\varphi,\psi)\rangle$$

imply $C(\langle\varphi\rangle,\langle\psi\rangle) = \langle(\varphi,\psi)\rangle$ by 6.31.

Proposition 6.37. $\langle[\varphi]\rangle = C[\langle\varphi\rangle C]$.

Proof. We have

$$(L, \varphi(L^2, LR)[\langle\varphi\rangle C]) = (L, \varphi LC[\langle\varphi\rangle C]) = (L, L\langle\varphi\rangle C[\langle\varphi\rangle C])$$
$$= (L^2, LR)[\langle\varphi\rangle C],$$

hence $[\varphi]L \leq (L^2, LR)[\langle\varphi\rangle C]$ by (££). Similarly,

$$(R, \langle\varphi\rangle C(RL, R^2)[\langle\varphi\rangle C]) = (R, R\langle\varphi\rangle C[\langle\varphi\rangle C]) = (RL, R^2)[\langle\varphi\rangle C]$$

gives $[\langle\varphi\rangle C]R \leq (RL, R^2)[\langle\varphi\rangle C]$.

Taking $\tau = (I, \varphi[\varphi]L, \langle\varphi\rangle C[\langle\varphi\rangle C]R)$, we get

$$(I, \langle\varphi\rangle C\tau) = (I, (\varphi L, \langle\varphi\rangle R)C\tau) = (I, \varphi(L^2, LR)\tau, \langle\varphi\rangle C(RL, R^2)\tau)$$
$$= (I, \varphi(L, \varphi[\varphi]L), \langle\varphi\rangle C(R, \langle\varphi\rangle C[\langle\varphi\rangle C]R)) = \tau,$$

hence $[\langle\varphi\rangle C] \leq \tau$. Therefore, $(L^2, LR)[\langle\varphi\rangle C] \leq [\varphi]L$ and $(RL, R^2)\circ [\langle\varphi\rangle C] \leq [\langle\varphi\rangle C]R$.

We conclude that $(L^2, LR)[\langle\varphi\rangle C] = [\varphi]L$ and $(RL, R^2)[\langle\varphi\rangle C] = [\langle\varphi\rangle C]R$, which implies $C[\langle\varphi\rangle C] = \langle[\varphi]\rangle$ by 6.31.

Proposition 6.38 (First Recursion Lemma). Let $\sigma = \Delta((L, \varphi\bar{2}), (\psi L, R^2))$. Then $\bar{1}[\sigma] = \mu\theta.(I, \varphi\theta\psi)$.

Proof. We have

$$(\psi, \sigma(\psi, R^2[\sigma])) = (\psi, \sigma(\psi L, R^2)[\sigma]) = (\psi, R\sigma[\sigma]) = (\psi, R^2[\sigma]),$$

hence $[\sigma]\psi \leq (\psi, R^2[\sigma])$. Therefore,

$$(I, \varphi\bar{1}[\sigma]\psi) \leq (I, \varphi\bar{1}(\psi, R^2[\sigma])) = (I, \varphi\bar{2}[\sigma]) = (L, \varphi\bar{2})[\sigma] = \bar{0}\sigma[\sigma] = \bar{1}[\sigma].$$

Let $(I, \varphi\tau\psi) \leq \tau$. Then the inequalities

$$(\psi L, R^2)(I, \Delta(\tau,\psi)) = (\psi, \Delta(\tau,\psi)\psi) = (I, \Delta(\tau,\psi))\psi,$$
$$((L, \varphi\bar{2})(I, \Delta(\tau,\psi)), \Delta(\tau,\psi)\psi) = ((I, \varphi\tau\psi), \Delta(\tau,\psi)\psi) \leq (\tau, \Delta(\tau,\psi)\psi)$$
$$= \Delta(\tau,\psi)$$

imply $\sigma(I, \Delta(\tau,\psi)) \leq \Delta(\tau,\psi)$ by 6.30, hence $R[\sigma] \leq \Delta(\tau,\psi)$ by 6.11. Therefore, $\bar{1}[\sigma] \leq \bar{0}\Delta(\tau,\psi) = \tau$, which completes the proof.

Proposition 6.39 (Second Recursion Lemma). Let $\varphi_1 = \varphi(\psi L, \chi R)$ and $\theta_1 = \mu\theta.(I, \varphi_1\theta\rho)$; the element θ_1 exists by 6.38. Then $\varphi_1\theta_1 = \mu\theta.\varphi(\psi, \chi\theta\rho)$.

Proof. We have

$$\varphi(\psi, \chi\varphi_1\theta_1\rho) = \varphi_1(I, \varphi_1\theta_1\rho) = \varphi_1\theta_1.$$

If $\varphi(\psi, \chi\tau\rho) \leq \tau$, then

$$(I, \varphi_1(I, \tau\rho)\rho) = (I, \varphi(\psi, \chi\tau\rho)\rho) \leq (I, \tau\rho),$$

hence $\theta_1 \leq (I, \tau\rho)$. Therefore,

$$\varphi_1\theta_1 \leq \varphi_1(I, \tau\rho) = \varphi(\psi, \chi\tau\rho) \leq \tau,$$

which completes the proof.

Proposition 6.40. Let $\rho = \Delta(L, R^2)$, $P = \Delta(\rho R, \rho)$, $\sigma = \Delta(L^2, A)$ and $Q = \mu\theta.\sigma(\theta R, L)$. (The element Q exists by 6.39.) Then $\langle\!\langle \varphi \rangle\!\rangle = P\langle \varphi \rangle Q$ for all φ.

Proof. It follows that

$$L\sigma = L^2, R^2\sigma = \sigma A^2, LA^2 = RL,$$

hence $\rho\sigma = \langle I \rangle L$ by 6.31. Therefore,

$$\rho R\sigma = \rho\sigma A = \langle I \rangle LA = \langle I \rangle R.$$

The equality $Q = \sigma(QR, L)$ multiplied on the left by ρR and ρ gives respectively $\rho RQ = \langle I \rangle L$, $\rho Q = QR$.

The equalities

$$L\langle \varphi \rangle = \varphi L = L\langle \varphi \rangle \rho, R^2\langle \varphi \rangle = \langle \varphi \rangle R^2, \langle \varphi \rangle \rho R^2 = \langle \varphi \rangle R\rho = R\langle \varphi \rangle \rho$$

imply $\rho\langle \varphi \rangle = \langle \varphi \rangle \rho$ by 6.31. Finally, we get

$$\rho R\langle \varphi \rangle Q = \langle \varphi \rangle \rho RQ = \langle \varphi \rangle L = L\langle\!\langle \varphi \rangle\!\rangle,$$
$$\rho\langle \varphi \rangle Q = \langle \varphi \rangle \rho Q = \langle \varphi \rangle QR, \langle\!\langle \varphi \rangle\!\rangle R = R\langle\!\langle \varphi \rangle\!\rangle,$$

hence $P\langle \varphi \rangle Q = \langle\!\langle \varphi \rangle\!\rangle$ by 6.31. The proof is complete.

Proposition 6.41. Let $D = \Delta(L^2, \langle R \rangle R)$. Then $\bar{n}D = \bar{n}\bar{n}$ for all n.

Proof.

$$\bar{n}D = L^2(\langle R \rangle R)^n = L^2\langle R \rangle^n R^n = LR^n LR^n.$$

Proposition 6.42. Let $\rho = C([I]L, R)$, $\sigma = D\langle \varphi \rangle \rho$ and $\tau = D\langle R\varphi \rangle \rho$. Then $\rho[\sigma] = ([\bar{0}\varphi], \rho[\tau])$.

Proof. Using the easy equalities

$$\sigma = (L\sigma, R\sigma) = (\bar{0}\varphi(\bar{0}, \bar{1}), \tau(\bar{0}, R^2)),$$

we get

$$(I, \bar{0}\varphi(\bar{0}, \bar{1})[\sigma]) = (I, \bar{0}\sigma[\sigma]) = (\bar{0}, \bar{1})[\sigma],$$

hence $[\bar{0}\varphi] \leq (\bar{0}, \bar{1})[\sigma]$. Similarly,

$$(I, \tau(\bar{0}, R^2)[\sigma]) = (I, R\sigma[\sigma]) = (\bar{0}, R^2)[\sigma]$$

implies $[\tau] \leq (\bar{0}, R^2)[\sigma]$.

On the other hand,

$$(I, \sigma(I, \bar{0}\varphi[\bar{0}\varphi], \tau[\tau])) = (I, \bar{0}\varphi(I, \bar{0}\varphi[\bar{0}\varphi]), \tau(I, \tau[\tau])) = (I, \bar{0}\varphi[\bar{0}\varphi], \tau[\tau])$$

implies $[\sigma] \leq (I, \bar{0}\varphi[\bar{0}\varphi], \tau[\tau])$, hence the above two inequalities are in fact equalities and

$$\rho[\sigma] = (L\rho[\sigma], R\rho[\sigma]) = ((\bar{0}, \bar{1})[\sigma], \rho(\bar{0}, R^2)[\sigma]) = ([\bar{0}\varphi], \rho[\tau]).$$

The proof is complete.

Proposition 6.43. Let $\rho = C([I]L, R)$ and $\Gamma = \lambda\theta.D\langle\theta\rangle\rho$. Then $\bar{n}\rho[\Gamma(\varphi)] = [\bar{n}\varphi]$ for all φ, n.

Proof. The previous statement gives that

$$\rho[\Gamma(\varphi)] = ([\bar{0}\varphi], \rho[\Gamma(R\varphi)])$$

for all φ, hence $\bar{0}\rho[\Gamma(\varphi)] = [\bar{0}\varphi]$. Supposing $\bar{n}\rho[\Gamma(\varphi)] = [\bar{n}\varphi]$ for all φ, we get

$$\overline{n+1}\,\rho[\Gamma(\varphi)] = \bar{n}\rho[\Gamma(R\varphi)] = [\bar{n}R\varphi] = [\overline{n+1}\,\varphi].$$

This completes the proof.

EXERCISES TO CHAPTER 6

Exercise 6.1. Prove the following equalities.

a. $A = LC[RL]\langle R\rangle[\langle L\rangle]$.
b. $(\varphi, \psi) = (RL, R)[\psi L^2][\varphi L]$.
c. $[\varphi] = \bar{1}[((\bar{0}, \varphi\bar{1}), \psi\bar{1})]$.
d. $\varphi[\psi] = R^2[((\bar{0}, \psi\bar{1}), \varphi\bar{1})]$.
e. $[\varphi]\psi[\chi] = R^2[(\chi(\bar{0}, \bar{1}), \psi(\bar{0}, \bar{1}), \varphi R^2)]$.
f. $(\varphi[\psi], \chi[\rho]) = (\varphi L, \chi R)[(L^2, \rho R)][\psi]$.
g. $[\varphi[\psi]] = R^2[((R^2, \psi\bar{1}), \bar{0}, \varphi\bar{1})]$.
h. $[\varphi[\psi]] = \bar{0}\bar{1}[(((\bar{0}, \varphi(R\bar{1}, R^2)), \bar{0}\bar{1}), \psi(R\bar{1}, R^2))]$.
i. $[\varphi[(\psi, \chi)]] = R^2\bar{1}[\sigma]$, where $\sigma = ((\psi\rho, R^2\bar{1}, \bar{0}, \varphi\rho), \chi\rho)$ and $\rho = (\bar{1}\bar{1}, \bar{0}\bar{1}, R^2)$.
j. $(\varphi, \psi) = \rho\Delta(R, \rho\Delta(\psi, L))\rho\Delta(I, \varphi)$, where $\rho = (\bar{0}, \bar{1})$.

Hint for assertion i. Show that

$$[(\psi, \chi)]R^2\bar{1}[\sigma] \leq \rho[\sigma], [\varphi[(\psi, \chi)]] \leq R^2\bar{1}[\sigma],$$
$$[\sigma] \leq (L, (\psi[(\psi, \chi)], I, I), \chi[(\psi, \chi)])[\varphi[(\psi, \chi)]].$$

Exercise 6.2 (Second Recursion Lemma, improved). Show that whenever $\psi\chi \leq \chi\chi_1$ and $\varphi(\chi, \tau\chi_1) \leq \tau$, then $(\mu\theta.\varphi(I, \theta\psi))\chi \leq \tau$.

Hint. Take σ as in 6.38, then $\mu\theta.\varphi(I, \theta\psi) = \varphi\bar{1}[\sigma]$ by 6.39. Show that $[\sigma]\chi \leq (\chi, \Delta((\chi, \tau\chi_1), \chi_1))$.

Ex 6.2

Exercise 6.3. $\Delta(\varphi, \psi) = \mu\theta.(\varphi, \varphi\psi, \theta\psi^2)$.

Hint. Writing θ_0 for $\mu\theta.(\varphi, \varphi\psi, \theta\psi^2)$, show that $\theta_0\psi \leq R\theta_0$ by exercise 6.2.

Exercise 6.4. Let $G = \Delta(\langle L\rangle, \langle R\rangle)$. Show that $\langle\langle I\rangle\rangle C = G(L, GR)$.

Hint. Show by 6.20 that $\langle L\rangle G = \langle I\rangle L$, hence $\langle L\rangle(L, GR) = L\langle\langle I\rangle\rangle C$;

Ex 6.3. [illegible] $\sigma = \Delta(L^2, A)$ [illegible]
$\langle I\rangle\sigma\langle I\rangle = \sigma$.
[illegible] 6.2 и (5).

similarly $\langle R\rangle G = GR$ and $\langle R\rangle(L, GR) = (L, GR)A^2$. The equality $\langle\!\langle I\rangle\!\rangle CA^2 = R\langle\!\langle I\rangle\!\rangle C$ completes the proof by 6.31.

Exercise 6.5. Establish the following assertions:

a. $P, Q \in cl(L, \langle\!\langle L\rangle\!\rangle, \langle\!\langle A\rangle\!\rangle/\circ, [\ \ ])$.

b. $P, Q \in cl(L, \langle\!\langle\!\langle L\rangle\!\rangle\!\rangle, \langle\!\langle\!\langle R\rangle\!\rangle\!\rangle/\circ, \Pi, [\ \ ])$.

Hint. Take $\sigma = \Delta(L^2, A)$, $\sigma_1 = \langle I\rangle\sigma\langle I\rangle$ and prove by exercises 6.3, 6.2 that $\sigma_1 = \sigma$, hence $Q = \mu\theta.\sigma_1(\theta R, L)$. Then show by 6.36–6.39 that

$$P, Q \in cl(L, \langle\!\langle I\rangle\!\rangle C, \langle\!\langle L\rangle\!\rangle, \langle\!\langle R\rangle\!\rangle, \langle\!\langle A\rangle\!\rangle/\circ, \Pi, [\ \ ]).$$

Make use of exercise 6.4, the equalities $\langle\!\langle R\rangle\!\rangle = \langle\!\langle L\rangle\!\rangle\langle\!\langle A\rangle\!\rangle$, $A = L^2\langle\!\langle A\rangle\!\rangle[L]^2$ and 6.9 to obtain assertion a.

Substitute $\langle I\rangle A$ for A in the above argument to show that

$$P, Q \in cl(L, \langle\!\langle I\rangle\!\rangle C, \langle\!\langle L\rangle\!\rangle, \langle\!\langle R\rangle\!\rangle, \langle\!\langle\!\langle I\rangle\!\rangle\!\rangle\langle\!\langle A\rangle\!\rangle/\circ, \Pi, [\ \ ]).$$

The equality $\langle A\rangle = C\langle I\rangle(\langle R\rangle, \langle RL\rangle)$ implies $\langle\!\langle\!\langle I\rangle\!\rangle\!\rangle\langle\!\langle A\rangle\!\rangle = \langle\!\langle\!\langle I\rangle\!\rangle\!\rangle\langle C\rangle\langle\!\langle I\rangle\!\rangle C(\langle\!\langle R\rangle\!\rangle, \langle\!\langle RL\rangle\!\rangle)$, hence

$$\langle\!\langle I\rangle\!\rangle C, \langle\!\langle\!\langle I\rangle\!\rangle\!\rangle\langle\!\langle A\rangle\!\rangle \in cl(L, \langle\!\langle\!\langle L\rangle\!\rangle\!\rangle, \langle\!\langle\!\langle R\rangle\!\rangle\!\rangle/\circ, \Pi, [\ \ ])$$

by

$$\begin{aligned}\langle\!\langle\!\langle I\rangle\!\rangle\!\rangle\langle C\rangle &= \langle G(L, GR)\rangle = \langle G\langle I\rangle(L, GR)\rangle\\ &= \langle G\rangle\langle\!\langle I\rangle\!\rangle C(\langle L\rangle, \langle G\rangle\langle R\rangle),\\ \langle G\rangle &= \langle\!\langle\!\langle L\rangle\!\rangle\!\rangle C[\langle\!\langle\!\langle R\rangle\!\rangle\!\rangle C].\end{aligned}$$

Exercise 6.6. Let $J = \lambda st.(2s+1)2^t$. Show that $\overline{s}\overline{t}P = \overline{J(s,t)}$ and $\overline{J(s,t)}Q = \overline{s}\overline{t}$ for all s, t.

Exercise 6.7.** Let $J: \omega^2 \to \omega$ and $\overline{s}\overline{t}\rho = \overline{J(s,t)}$, $\overline{J(s,t)}\sigma = \overline{s}\overline{t}$ for all s, t. Show that $\langle\!\langle \varphi\rangle\!\rangle = \langle\!\langle I\rangle\!\rangle\rho\langle\varphi\rangle\sigma$ for all φ.

Hint. Use 5.9**.

Axiom μA_1 ensures that the element $I^* = \mu\theta.(L, \langle\theta\rangle R)$ exists and has certain additional properties.

Exercise 6.8*. Prove that

$$\begin{aligned}(£)^*\quad &(L, \langle I^*\rangle R) \leq I^*,\\ &\langle I\rangle R\psi \leq \langle\psi\rangle\psi_1 \,\&\, (L\psi, \langle\tau\rangle\psi_1) \leq \tau \Rightarrow I^*\psi \leq \tau,\\ &C(\langle L\rangle, P\tau Q\langle R\rangle) \leq \tau \Rightarrow \langle I^*\rangle \leq \tau.\end{aligned}$$

Hint. Supposing $\langle I\rangle R\psi \leq \langle\psi\rangle\psi_1$ and $(L\psi, \langle\tau\rangle\psi_1) \leq \tau$, show that the simple segment $\{\theta/\theta\psi \leq \tau\}$ is closed under $\lambda\theta.(L, \langle\theta\rangle R)$. The element $\sigma = \mu\theta.C(\langle L\rangle, P\theta Q\langle R\rangle)$ exists by 6.39; show that the simple segment $\{\theta/\langle\theta\rangle \leq \sigma\}$ is closed under $\lambda\theta.(L, \langle\theta\rangle R)$.

Exercise 6.9. Let I^* meet $(£)^*$ and $\sigma = \mu\theta.C(\langle L\rangle, P\theta Q\langle R\rangle)$. Prove that $I^* = L\sigma(I, I)$.

Hint. It suffices to show that $\langle I^*\rangle = \sigma$. The inequality $C(\langle L\rangle,$

$P\langle I^*\rangle Q\langle R\rangle) \leq \langle I^*\rangle$ implies $\sigma \leq \langle I^*\rangle$ while, on the other hand, the last implication of (£)* ensures that $\langle I^*\rangle \leq \sigma$.

It is the case that $I^* = \langle I\rangle = I$ in example 4.1 and $I^* < \langle I\rangle < I$ in example 4.3.

Exercise 6.10. Let I^* meet (£)* and $\Gamma:\mathscr{F}^2 \to \mathscr{F}$ such that $\Gamma(\varphi,\psi) = (\varphi, \langle\Gamma(\varphi,\psi)\rangle\psi)$ for all φ,ψ. Show that the operation $\Delta^* = \lambda\varphi\psi.\mu\theta.(\varphi,\langle\theta\rangle\psi)$ exists and satisfies the following analogue to 6.30: if $\langle I\rangle\psi\chi \leq \langle\chi\rangle\chi_1$, $(\varphi\chi, \langle\tau\rangle\chi_1) \leq \tau$, then $\Delta^*(\varphi,\psi)\chi \leq \tau$.

Hint. Take $\Delta^* = \lambda\varphi\psi.I^*\Gamma(\varphi,\psi)$ by definition.

Exercise 6.11. Prove that the operation $\circ$ cannot be expressed in terms of Π, $\langle\ \ \rangle, [\ \ ]$ and members of $\mathscr{D}$, i.e. there is no expression $\mathscr{V}(\theta_1,\theta_2)$ constructed from θ_1,θ_2 and members of $\mathscr{D}$ using $\Pi, \langle\ \ \rangle, [\ \ ]$ such that $\varphi\psi = \mathscr{V}(\varphi,\psi)$ for all φ,ψ.

Hint. Suppose there is such an expression $\mathscr{V}$. Define $lh(I) = 0$, $lh(\alpha L) = lh(\alpha R) = lh(\alpha) + 1$. Let $\mathscr{D}_0$ be the finite set consisting of L and all the members of $\mathscr{D}$ to occur in $\mathscr{V}$. Take $n > lh(\alpha)$ for all $\alpha\in\mathscr{D}_0$. Show that if $\varphi\in cl(\mathscr{D}_0/\Pi, \langle\ \ \rangle, [\ \ ]) \cap \mathscr{D}$, then $lh(\varphi) < n$. Therefore, $L^n \notin cl(\mathscr{D}_0/\Pi, \langle\ \ \rangle, [\ \ ])$, which is a contradiction.

Exercise 6.12. Prove that the operation Δ cannot be expressed in terms of $\circ$, Π, $\langle\ \ \rangle$ and I, L, R.

Hint. Take $\mathscr{N} = \{x/x = \bar{k}_1\ldots\bar{k}_n \& n, k_1,\ldots,k_n\in\omega\}$ $(= \mathscr{D}\setminus R\mathscr{D})$ and define $lh(I) = 0$, $lh(x\bar{n}) = lh(x) + 1$. Show that if $\varphi\in cl(I,L,R/\circ,\Pi,\langle\ \ \rangle)$, then $\exists n\forall x(lh(x) = n \Rightarrow x\varphi\in\mathscr{N})$. Therefore, $\Delta(I,(I,I)) \notin cl(I,L,R/\circ,\Pi,\langle\ \ \rangle)$.

Exercise 6.13. Show that the operation $[\ \ ]$ can not be expressed in terms of $\circ, \Pi, \Delta$ and I, L, R.

Hint. Take $\mathscr{N}$ as above and show that if $\varphi\in cl(I,L,R/\circ,\Pi,\Delta)$, then $\forall x\exists y(yx\varphi\in\mathscr{N})$. Therefore, $[R] \notin cl(I,L,R/\circ,\Pi,\Delta)$.

We give a detailed proof of the following Translation Independence Theorem since it introduces an important new technique called the *unwinding method*.

Exercise 6.14*.** Show that the operation $\langle\ \ \rangle$ can not be expressed in terms of $\circ, \Pi, [\ \ ]$ and I, L, R.

Hint. By way of contradiction, take $\alpha\in\mathscr{D}\setminus\{I\}$ and suppose $\langle\alpha\rangle\in cl(I,L,R/\circ,\Pi,[\ \ ])$. Propositions 6.8, 6.9, 6.15, 6.16 imply that $\langle\alpha\rangle = \bar{I}[\varphi]$ for a certain $\varphi\in cl(I,L,R/\circ,\Pi)$. Let $m, \varphi_0,\ldots,\varphi_m$ correspond to φ by 4.12 and $l > \max\{k/\exists i \leq m(\alpha\bar{k} = L\varphi_i[\varphi])\}$. (The last set is finite by 4.4.)

For each β such that $R^l\langle\alpha\rangle = \beta\varphi[\varphi]$ we construct a β_1 such that $\beta\varphi = \beta_1 R$ and $R^l\langle\alpha\rangle = \beta_1\varphi[\varphi]$. It follows from 4.12 that either $\beta\varphi\in\mathscr{D}$ or $\beta\varphi = \varphi_i$ for a certain $i \leq m$. The latter implies $R^l\langle\alpha\rangle = \beta\varphi[\varphi] = \varphi_i[\varphi]$; hence $\alpha\bar{l} = L\varphi_i[\varphi]$ contrary to the choice of l. Therefore, $\beta\varphi\in\mathscr{D}$. Supposing $\beta\varphi = I$, we get $\alpha\bar{l} = L\beta\varphi[\varphi] = I$ contrary to 4.4. Suppose that $\beta\varphi = \beta_1 L$. Then $R^l\langle\alpha\rangle = \beta_1 L[\varphi] = \beta_1$; hence $\langle\alpha\rangle\in\mathscr{D}$, which leads easily to a contradiction. Therefore,

$\beta\varphi = \beta_1 R$ for some β_1, namely $\beta_1 = \beta\varphi(I, I)$. It also follows that $R^I\langle\alpha\rangle = \beta\varphi[\varphi] = \beta_1 R[\varphi] = \beta_1\varphi[\varphi]$.

Take $\beta_0 = R^I L$. Then $\langle\alpha\rangle = \bar{1}[\varphi]$ implies $R^I\langle\alpha\rangle = \beta_0\varphi[\varphi]$ and we obtain by the construction suggested above an infinite sequence $\{\beta_n\}$ such that $\beta_n\varphi = \beta_{n+1}R$ for all n. The regular segment $\mathscr{E} = \{\theta/\forall n(\beta_n R\theta \le O)\}$ is closed under $\lambda\theta.(I, \varphi\theta)$, hence $[\varphi] \in \mathscr{E}$. Therefore, $\alpha\bar{1} = L\beta_0 R[\varphi] \le LO = O$, which is not the case. Thus the proof is complete.

It is worth mentioning that the above argument depends on the following corollary to the axiom μA_3. For any sequence $\{\alpha_n\}$,

(@@) (£££) $\forall n(\alpha_n\varphi = \alpha_{n+1}R) \Rightarrow \alpha_0 R[\varphi] = O.$ Hereditary

CHAPTER 7

Relative recursiveness

In this chapter we introduce several abstract concepts of effective computability in IOS. The basic notion is that of relative recursiveness, although the others also play an important role. Some simple properties and characterizations of these notions are established. Both elements and mappings are studied.

Given an IOS $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$, a subset $\mathscr{B}$ of $\mathscr{F}$ and an element $\varphi \in \mathscr{F}$, we say that

φ is *polynomial in* $\mathscr{B}$ iff
$\varphi \in cl(\{L, R\} \cup \mathscr{B}/\circ, \Pi)$,
φ is *primitive in* $\mathscr{B}$ iff
$\varphi \in cl(\{L, R\} \cup \mathscr{B}/\circ, \Pi, \langle\ \ \rangle)$,
φ is *primitive recursive in* $\mathscr{B}$ iff
$\varphi \in cl(\{L, R\} \cup \mathscr{B}/\circ, \Pi, \Delta)$,
φ is *primitive recursive in* $\mathscr{B}$ iff
$\varphi \in cl(\{L, R\} \cup \mathscr{B}/\circ, \Pi, [\ \ ])$,
φ is *recursive in* $\mathscr{B}$ iff
$\varphi \in cl(\{L, R\} \cup \mathscr{B}/\circ, \Pi, \langle\ \ \rangle, [\ \ ])$.

Our notion of prime recursiveness corresponds to the notion of recursiveness considered in Georgieva [1980].

The elements polynomial (primitive, primitive recursive, prime recursive, recursive) in $\varnothing$ are called *polynomial* (respectively *primitive, primitive recursive, prime recursive, recursive*). For example, the elements A, B introduced in the previous chapter are polynomial, while C, D, G, P are primitive recursive and I, O, Q are recursive.

Proposition 7.1. If φ is polynomial in $\mathscr{B}$, then φ is both primitive in $\mathscr{B}$ and prime recursive in $\mathscr{B}$. If φ is primitive in $\mathscr{B}$, then φ is primitive recursive in $\mathscr{B}$. If φ is primitive recursive in $\mathscr{B}$ or prime recursive in $\mathscr{B}$, then φ is recursive in $\mathscr{B}$.

This follows from the definitions, using 6.32.

Proposition 7.2. If φ is polynomial (respectively primitive, primitive recursive, prime recursive, recursive) in $\mathscr{B}$ and $\mathscr{B} \subseteq \mathscr{B}_1$, then φ is polynomial (primitive etc.) in $\mathscr{B}_1$.

This follows from the corresponding definitions.

Proposition 7.3. Let φ be polynomial (primitive etc.) in $\mathscr{B}$. If all the members of $\mathscr{B}$ are polynomial (primitive etc.) in $\mathscr{B}_1$, then so is φ.

This follows by a trivial induction on the construction of φ.

Proposition 7.4. If φ is polynomial (primitive etc.) in $\mathscr{B}$, then φ is polynomial (primitive etc.) in a finite subset of $\mathscr{B}$.

Proposition 7.5. An element φ is prime recursive in $\mathscr{B}$ iff

$$\varphi \in cl(\{L, A\} \cup \mathscr{B}/\circ, [\quad]),$$

while φ is recursive in $\mathscr{B}$ iff

$$\varphi \in cl(\{L, A\} \cup \mathscr{B}/\circ, \langle \quad \rangle, [\quad]).$$

This follows from 6.9 and the equality $R = LA$.

Proposition 7.6. An element φ is primitive recursive in $\{I\} \cup \mathscr{B}$ iff $\varphi \in cl(\{R, B, I\} \cup \mathscr{B}/\circ, \Delta)$.

This follows from 6.29 and the equality $L = RB$.

Proposition 7.7. An element φ is recursive in $\mathscr{B}$ iff

$$\varphi \in cl(\{R, B\} \cup \mathscr{B}/\circ, \langle \quad \rangle, [\quad]).$$

This follows from 5.12, 6.29.

Proposition 7.8. If φ is primitive in $\mathscr{B}$, then

$$\langle \varphi \rangle \in cl(\{C, P, Q, \langle L \rangle, \langle R \rangle\} \cup \langle \mathscr{B} \rangle/\circ, \Pi)$$

(where $\langle \mathscr{B} \rangle$ stands for $\{\langle \psi \rangle / \psi \in \mathscr{B}\}$).

Proof. By induction on the construction of φ.

If $\varphi \in \{L, R\} \cup \mathscr{B}$, then $\langle \varphi \rangle \in \{\langle L \rangle, \langle R \rangle\} \cup \langle \mathscr{B} \rangle$.

If φ, ψ have the required property, then so do $\varphi\psi$, (φ, ψ) and $\langle \varphi \rangle$ since $\langle \varphi\psi \rangle = \langle \varphi \rangle \langle \psi \rangle$, $\langle (\varphi, \psi) \rangle = C(\langle \varphi \rangle, \langle \psi \rangle)$ and $\langle\langle \varphi \rangle\rangle = P\langle \varphi \rangle Q$.

Proposition 7.9. If φ is primitive in $\{I\} \cup \mathscr{B}$, then φ is polynomial in $\{I, C, P, Q, \langle L \rangle, \langle R \rangle\} \cup \langle \mathscr{B} \rangle$.

This follows from 7.8 since $\varphi = L\langle \varphi \rangle (I, I)$.

Proposition 7.10. If φ is recursive in $\mathscr{B}$, then

$$\langle \varphi \rangle \in cl(\{C, P, Q, \langle L \rangle, \langle R \rangle\} \cup \langle \mathscr{B} \rangle/\circ, \Pi, [\quad]).$$

Proof. We complete the proof of 7.8 by considering one more case. Namely, whenever φ has the required property, then so does $[\varphi]$ since $\langle [\varphi] \rangle = C[\langle \varphi \rangle C]$.

Proposition 7.11 (Pull Back Theorem). The following are equivalent.

(1) φ is recursive in $\mathscr{B}$.

(2) $\varphi \in cl(\{L, C, P, Q, \langle L \rangle, \langle R \rangle\} \cup \langle \mathscr{B} \rangle/\circ, [\quad]).$

(3) $\varphi \in cl(\{L, \langle B \rangle, \langle\!\langle L \rangle\!\rangle, \langle\!\langle A \rangle\!\rangle\} \cup \langle \mathscr{B} \rangle / \circ, [\ \ ])$.
(4) $\varphi \in cl(\{L, \langle C \rangle, \langle\!\langle L \rangle\!\rangle, \langle\!\langle R \rangle\!\rangle\} \cup \langle \mathscr{B} \rangle / \circ, [\ \ ])$.

Proof. The implication (4)$\Rightarrow$(1) is immediate. We shall prove (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4).

The implication (1)$\Rightarrow$(2) follows from 7.10 since $\varphi = L\langle \varphi \rangle [L]$; the operation Π is omitted by 6.9 and exercise 6.1a.

Assume (2). The equalities $C = \langle B \rangle^2 [\langle A \rangle^2]$, $\langle A \rangle = L \langle\!\langle A \rangle\!\rangle [L]$ and exercise 6.5a give

$$C, P, Q \in cl(L, \langle B \rangle, \langle\!\langle L \rangle\!\rangle, \langle\!\langle A \rangle\!\rangle / \circ, [\ \ ]),$$

which implies (3).

Suppose (3). We have

$$\langle B \rangle = \langle (LR, L) \rangle = C(\langle L \rangle \langle R \rangle, \langle L \rangle),$$
$$\langle A \rangle = \langle (R, RL) \rangle = C(\langle R \rangle, \langle RL \rangle).$$

Using 6.9, we get

$$\langle\!\langle A \rangle\!\rangle = \langle C \rangle C(\langle\!\langle R \rangle\!\rangle, \langle\!\langle RL \rangle\!\rangle) = \langle C \rangle CA[\langle\!\langle R \rangle\!\rangle L^2][\langle\!\langle RL \rangle\!\rangle L],$$

which implies (4) by exercise 6.1a. This completes the proof.

We now construct several primitive recursive elements to be used subsequently.

Proposition 7.12. Let $H = \Delta(\langle I \rangle, \langle I \rangle)$. Then $\bar{n}H = \langle I \rangle$ for all n.

Proof. H is primitive recursive since $\langle I \rangle = \Delta(L, R)$. It follows that $\bar{n}H = \langle I \rangle \langle I \rangle^n = \langle I \rangle$.

Proposition 7.13. $\bar{n}G = \langle \bar{n} \rangle$ for all n.

Proof.

$$\bar{n}G = \langle L \rangle \langle R \rangle^n = \langle LR^n \rangle = \langle \bar{n} \rangle.$$

Proposition 7.14. Let $T = \langle C \rangle C(H, GH)$. Then $\bar{m}\bar{n}T = (\bar{m}, \bar{n})$ for all m, n.

Proof.

$$\bar{m}\bar{n}T = \bar{m}C\bar{n}C(H, GH) = (\bar{m}\bar{n}H, \bar{m}\bar{n}GH) = (\bar{m}\langle I \rangle, \bar{m}\langle \bar{n} \rangle H)$$
$$= (\bar{m}, \bar{n}\langle I \rangle) = (\bar{m}, \bar{n}).$$

It is convenient to write $\langle \varphi \rangle_0$ for φ and $\langle \varphi \rangle_{n+1}$ for $\langle\!\langle \varphi \rangle\!\rangle_n$. Notice that whenever $m < n$, then $\langle R \rangle_m \langle \varphi \rangle_n = \langle \varphi \rangle_n \langle R \rangle_m$. Moreover, $\bar{s}_1 \ldots \bar{s}_n \langle \varphi \rangle_n = \varphi \bar{s}_1 \ldots \bar{s}_n$ for all $n, s_1, \ldots, s_n$.

Proposition 7.15. Let $G_0 = \langle I \rangle$ and $G_{n+1} = G_n \langle G \rangle_n$. Then $\bar{m}G_n = \langle \bar{m} \rangle_n$ for all m, n.

Proof. We have

$$\bar{m}G_0 = \bar{m} = \langle \bar{m} \rangle_0, \bar{m}G_{n+1} = \bar{m}G_n \langle G \rangle_n = \langle \bar{m} \rangle_n \langle G \rangle_n$$
$$= \langle \bar{m}G \rangle_n = \langle\!\langle \bar{m} \rangle\!\rangle_n = \langle \bar{m} \rangle_{n+1}.$$

Proposition 7.16. Let $D_1 = D$ and $D_{n+1} = \langle D_n \rangle D \langle G_n \rangle$ for $n > 0$. Then $\overline{s_1} \dots \overline{s_n} D_n = \overline{s_1} \dots \overline{s_n} \overline{s_1} \dots \overline{s_n}$ for all $s_1, \dots, s_n$.

Proof. Proposition 6.41 gives $\overline{s_1} D_1 = \overline{s_1}\,\overline{s_1}$, while $\overline{s_1} \dots \overline{s_{n+1}} D_{n+1} = \overline{s_1} \dots \overline{s_n} \overline{s_1} \dots \overline{s_{n+1}} G_n \overline{s_{n+1}} = \overline{s_1} \dots \overline{s_n} \overline{s_1} \dots \overline{s_n} \langle \overline{s_{n+1}} \rangle_n \overline{s_{n+1}} = \overline{s_1} \dots \overline{s_{n+1}} \overline{s_1} \dots \overline{s_{n+1}}$.

Proposition 7.17. Let $C_1 = C$ and $C_{n+1} = \langle C_n \rangle C$ for $n > 0$. Then $\overline{s_1} \dots \overline{s_n} C_n = (\overline{s_1} \dots \overline{s_n} L, \overline{s_1} \dots \overline{s_n} R)$ for all $s_1, \dots, s_n$.

Proof. Proposition 6.35 gives $\overline{s_1} C_1 = (\overline{s_1} L, \overline{s_1} R)$, while $\overline{s_1} \dots \overline{s_{n+1}} C_{n+1} = \overline{s_1} \dots \overline{s_n} C_n \overline{s_{n+1}} C = (\overline{s_1} \dots \overline{s_n} L, \overline{s_1} \dots \overline{s_n} R)(\overline{s_{n+1}} L, \overline{s_{n+1}} R) = (\overline{s_1} \dots \overline{s_{n+1}} L, \overline{s_1} \dots \overline{s_{n+1}} R)$, completing the proof.

A notion of relative recursiveness for mappings is introduced by parametrizing the corresponding notion for elements. Namely, the unary mappings Γ *recursive in* $\mathscr{B}$ are defined inductively as follows.

1. The mappings $\Gamma = \lambda\theta.\theta$ and $\Gamma = \lambda\theta.\psi$, $\psi \in \{L, R\} \cup \mathscr{B}$, are recursive in $\mathscr{B}$.

2. If $\Gamma_1, \Gamma_2 : \mathscr{F} \to \mathscr{F}$ are recursive in $\mathscr{B}$, then so are $\Gamma = \lambda\theta.\Gamma_1(\theta)\Gamma_2(\theta)$, $\Gamma = \lambda\theta.(\Gamma_1(\theta), \Gamma_2(\theta))$, $\Gamma = \lambda\theta.\langle \Gamma_1(\theta) \rangle$ and $\Gamma = \lambda\theta.[\Gamma_1(\theta)]$. In other words, $\Gamma : \mathscr{F} \to \mathscr{F}$ is recursive in $\mathscr{B}$ iff for all θ the element $\Gamma(\theta)$ is uniformly recursive in $\{\theta\} \cup \mathscr{B}$.

To define the n-ary mappings $\Gamma : \mathscr{F}^n \to \mathscr{F}$ recursive in $\mathscr{B}$, just take $\Gamma = \lambda\theta_1 \dots \theta_n.\theta_i$, $1 \le i \le n$, and $\Gamma = \lambda\theta_1 \dots \theta_n.\psi$, $\psi \in \{L, R\} \cup \mathscr{B}$, in the first clause above.

Proposition 6.9 implies an equivalent definition with A substituted for R in the first clause and the case of Π dropped in the second one. Other equivalent definitions can be obtained by using 6.29 and (a parametrized version of) 7.11.

In particular, one may take the mappings $\Gamma = \lambda\theta_1 \dots \theta_n.\langle \theta_i \rangle$, $1 \le i \le n$, and $\Gamma = \lambda\theta_1 \dots \theta_n.\psi$, $\psi \in \mathscr{B}_0 \cup \langle \mathscr{B} \rangle$, in the first clause, omitting the cases of Π, $\langle \quad \rangle$ in the second clause, where $\mathscr{B}_0$ is a set of primitive elements such that $cl(\mathscr{B}_0/\circ, [\quad])$ is the set of all recursive elements. (Therefore, $cl(\mathscr{B}_0 \cup \langle \mathscr{B} \rangle/\circ, [\quad])$ is exactly the set of all the elements recursive in $\mathscr{B}$.) For instance, $\mathscr{B}_0 = \{L, \langle B \rangle, \langle L \rangle_2, \langle A \rangle_2\}$ would do by 7.11. The set $\mathscr{B}_0$ may be assumed finite without loss of generality. Actually,

$$L, \langle B \rangle, \langle L \rangle_2, \langle A \rangle_2 \in cl(\mathscr{B}_0/\circ, [\quad]),$$

hence there is a finite subset $\mathscr{B}_1$ of $\mathscr{B}_0$ such that

$$L, \langle B \rangle, \langle L \rangle_2, \langle A \rangle_2 \in cl(\mathscr{B}_1/\circ, [\quad]).$$

Therefore, $\mathscr{B}_0$ may be replaced by $\mathscr{B}_1$.

Notions of mappings *polynomial*, *primitive*, *primitive recursive* and *prime recursive in* $\mathscr{B}$ are introduced in a similar way, while certain equivalent definitions are implied 7.5, 7.6. If $\mathscr{B} = \varnothing$, then one has absolute recursiveness, prime recursiveness etc.

The elementary properties 7.1–7.4 hold for mappings as well. Three properties germane to mappings follow.

Proposition 7.18. If $\Gamma_0 : \mathscr{F}^m \to \mathscr{F}$ and $\Gamma_1, \ldots, \Gamma_m : \mathscr{F}^n \to \mathscr{F}$ are recursive in $\mathscr{B}$, then so is

$$\Gamma = \lambda\theta_1 \ldots \theta_n . \Gamma_0(\Gamma_1(\theta_1, \ldots, \theta_n), \ldots, \Gamma_m(\theta_1, \ldots, \theta_n)).$$

This follows by an easy induction on the construction of Γ_0.

Proposition 7.19. A n-ary mapping Γ is recursive in $\psi_1, \ldots, \psi_m$ iff there is a $m+n$-ary recursive mapping Γ^* such that

$$\Gamma = \lambda\theta_1 \ldots \theta_n . \Gamma^*(\psi_1, \ldots, \psi_m, \theta_1, \ldots, \theta_n).$$

Proof. The 'if'-part follows by 7.18. The 'only if'-part is proved by a trivial induction on the construction of Γ.

Therefore, in studying n-ary mappings it suffices to confine attention to absolutely recursive mappings. On the other hand, one may consider only *unary* mappings when studying the relative recursiveness, because a pairing scheme Π, L, R for $\mathscr{F}$ is available.

Proposition 7.20. Let $\Gamma : \mathscr{F}^n \to \mathscr{F}$, $n > 1$, and

$$\Gamma^* = \lambda\theta . \Gamma(\bar{0}\theta, \ldots, \overline{n-2}\theta, R^{n-1}\theta).$$

Then $\Gamma = \lambda\theta_1 \ldots \theta_n . \Gamma^*((\theta_1, \ldots, \theta_n))$ and Γ is recursive in $\mathscr{B}$ iff so is Γ^*.

This follows from 7.18.

The following Transition Theorem shows that all the unary mappings recursive in $\mathscr{B}$ have the *transition property* established in 6.42 for [ ].

Proposition 7.21. If Γ is a unary mapping recursive in $\mathscr{B}$, then there is a mapping Γ^* recursive in $\mathscr{B}$ such that $\Gamma^*(\theta) = (\Gamma(L\theta), \Gamma^*(R\theta))$ for all θ. In particular, $\bar{n}\Gamma^*(\theta) = \Gamma(\bar{n}\theta)$ for all θ, n.

Proof. By induction on the construction of Γ.

Let $\Gamma = \lambda\theta . \langle\theta\rangle$. Taking $\Gamma^* = \lambda\theta . G\langle\theta\rangle$, we get

$$\Gamma^*(\theta) = (\langle L\rangle, G\langle R\rangle)\langle\theta\rangle = (\langle L\theta\rangle, G\langle R\theta\rangle) = (\Gamma(L\theta), \Gamma^*(R\theta)).$$

Let $\Gamma = \lambda\theta . \psi$, $\psi \in \mathscr{B}_0 \cup \langle\mathscr{B}\rangle$. Taking $\Gamma^* = \lambda\theta . \Delta(\psi, I)$, we get

$$\Gamma^*(\theta) = (\psi, \Delta(\psi, I)) = (\Gamma(L\theta), \Gamma^*(R\theta)).$$

Let Γ_1^*, Γ_2^* correspond to Γ_1, Γ_2.

If $\Gamma = \lambda\theta . \Gamma_1(\theta)\Gamma_2(\theta)$, then take $\Gamma^* = \lambda\theta . D\langle\Gamma_1^*(\theta)\rangle\Gamma_2^*(\theta)$. It follows that

$$\begin{aligned}\Gamma^*(\theta) &= (L^2, D\langle R\rangle R)\langle\Gamma_1^*(\theta)\rangle\Gamma_2^*(\theta)\\ &= (L\Gamma_1^*(\theta)L\Gamma_2^*(\theta), D\langle R\Gamma_1^*(\theta)\rangle R\Gamma_2^*(\theta))\\ &= (\Gamma_1(L\theta)\Gamma_2(L\theta), D\langle\Gamma_1^*(R\theta)\rangle\Gamma_2^*(R\theta)) = (\Gamma(L\theta), \Gamma^*(R\theta)).\end{aligned}$$

If $\Gamma = \lambda\theta . [\Gamma_1(\theta)]$, then take $\Gamma^* = \lambda\theta . \rho[D\langle\Gamma_1^*(\theta)\rangle\rho]$, where $\rho = C([I]L, R)$. Using 6.42, we get

$$\begin{aligned}\Gamma^*(\theta) &= ([L\Gamma_1^*(\theta)], \rho[D\langle R\Gamma_1^*(\theta)\rangle\rho]) = ([\Gamma_1(L\theta)], \rho[D\langle\Gamma_1^*(R\theta)\rangle\rho])\\ &= (\Gamma(L\theta), \Gamma^*(R\theta)),\end{aligned}$$

which completes the proof.

Notice that the mapping Γ^* constructed in 7.21 also satisfies the equality $\langle I \rangle \Gamma^*(\theta) = \Gamma^*(\theta)$ for all θ.

EXERCISES TO CHAPTER 7

Exercise 7.1. Let I^* satisfy (£)*. Prove that I^* is recursive.

Hint. Use exercise 6.9 and propositions 6.39, 6.38.

Exercise 7.2 (Modification Lemma). Let $W, L_1, R_1 \in \mathscr{F}$ and $L_1 W = L$, $R_1 W = R$. (In particular, whenever Π_1, L_1, R_1 is another pairing scheme for $\mathscr{F}$, then $W = (L, R)_1$ and L_1, R_1 satisfy these assumptions.) Taking $(\varphi, \psi)_1 = W(\varphi, \psi)$ show that $\mathscr{S}_1 = (\mathscr{F}, I, \Pi_1, L_1, R_1)$ is an IOS and φ is recursive (prime recursive) in $\{L_1, R_1, (L, R)_1\} \cup \mathscr{B}$ iff φ is recursive$_1$ (prime recursive$_1$) in $\{L, R, (L_1, R_1)\} \cup \mathscr{B}$.

Hint. Show that $\langle \quad \rangle_1 = \lambda\varphi . \mu\theta . (\varphi L_1 L, R)_1 (I, \theta R_1)$ satisfies (£) by exercise 6.2 and $[\quad]_1 = \lambda\varphi . (L, R)_1 [\varphi (L, R)_1]$ satisfies (££), then use exercise 4.6.

Exercise 7.3. Assuming $\langle I \rangle = I$ (cf. the comments to 6.22), show that the following are equivalent:

(1) φ is recursive in $\mathscr{B}$.
(2) $\varphi \in cl(\{L, P, Q, \langle L \rangle, \langle R \rangle\} \cup \langle \mathscr{B} \rangle / \circ, \Pi, [\quad])$.
(3) $\varphi \in cl(\{L, \langle L \rangle_2, \langle A \rangle_2\} \cup \langle \mathscr{B} \rangle / \circ, [\quad])$.
(4) $\varphi \in cl(\{L, \langle L \rangle_3, \langle R \rangle_3\} \cup \langle \mathscr{B} \rangle / \circ, \Pi, [\quad])$.

Hint. Use 7.11, exercise 6.4, the equality $\langle B \rangle = C(\langle LR \rangle, \langle L \rangle)$ and exercise 6.5b.

Exercise 7.4. Show that propositions 7.18–7.20 remain valid with 'prime recursive' (or 'primitive recursive', 'primitive', 'polynomial') substituted for 'recursive'.

The following exercise establishes a Parametrized Transition Theorem.

Exercise 7.5. Let Γ be a binary mapping recursive in $\mathscr{B}$. Prove that there is a mapping Γ^* recursive in $\mathscr{B}$ such that

$$\Gamma^*(\theta, \theta_1) = (\Gamma(\theta, L\theta_1), \Gamma^*(\theta, R\theta_1)) \quad \text{for all } \theta, \theta_1.$$

Hint. Let $\Gamma_1 = \lambda\theta . \Gamma(L\theta, R\theta)$ and Γ_1^* correspond to Γ_1 by 7.21. Then take $\Gamma^* = \lambda\theta\theta_1 . \Gamma_1^*(C(\Delta(\theta, I), \theta_1))$.

Exercise 7.6. Construct a binary primitive recursive mapping Δ^* such that $\Delta^*(\theta_1, \theta_2) = (\Delta(L\theta_1, L\theta_2), \Delta^*(R\theta_1, R\theta_2))$ for all θ_1, θ_2. Using Δ^*, show that the Transition Theorem holds for relative primitive recursiveness, provided I is added to the initial elements.

Hint. Take $\Delta^* = \lambda\theta_1\theta_2 . G(\theta_1, \Delta(D\langle\theta_1\rangle, D\langle\theta_2\rangle)\theta_2)$.

The next six exercises study 'multiple-valued' spaces (e.g. examples 4.8 but not 4.1, 4.3, 4.6, 4.7), in which L, R have an upper bound U. The element U will make it possible to discribe relative partial recursiveness and Friedman's computability by effective definitional schemes. Abstract partial recursiveness in spaces with elements U is studied in greater detail in Ivanov [1987].

Exercise 7.7. Let $L, R \leq U$. Construct an element $\bar{\omega}$ recursive in U such that $\bar{n} \leq \bar{\omega}$ for all n.

Hint. Take $\bar{\omega} = \bar{1}[\rho]$, where $\rho = \langle U \rangle C(L, R^2)$. Using the inequalities $\varphi = L(\varphi, \psi) \leq U(\varphi, \psi)$ and $\psi \leq U(\varphi, \psi)$, show that $\bar{n} \leq \overline{n+1}[\rho]$ and $\overline{n+2}[\rho] \leq \overline{n+1}[\rho]$, hence $\bar{n} \leq \overline{n+1}[\rho] \leq \bar{1}[\rho]$.

Exercise 7.8. Show that the following are equivalent.

(1) $L, R \leq U$, $U(I, I) \leq I$.
(2) $U(\varphi, \psi) = \sup\{\varphi, \psi\}$ for all φ, ψ.

Exercise 7.9*.** Let $L, R \leq U$ and $U(I, I) \leq I$. Construct an element $\bar{\omega}$ recursive in U such that $\bar{\omega}\varphi = \sup_n \bar{n}\varphi$ for all φ.

Hint. Take $\bar{\omega}$ as in exercise 7.7. Supposing $\bar{n}\varphi \leq \tau$ for all n, show that $[\rho] \in \{\theta / L\theta \leq I \,\&\, \forall n (\overline{n+1}\,\theta\varphi \leq \tau)\}$.

Remark. This exercise ensures that whenever $\mathscr{U}$ is a subset of $\mathscr{F}$ of the form $\{\bar{n}\sigma / n \in \omega\}$, then $\bar{\omega}\sigma = \sup \mathscr{U}$, e.g. $\bar{\omega} = \sup_n \bar{n}$, $\bar{\omega}\Delta(I, \varphi) = \sup_n \varphi^n$, $\bar{\omega}\Delta(\varphi, \psi)\chi = \sup_n \varphi\psi^n\chi$ etc.

Exercise 7.10. Prove that the following are equivalent.

(1) $L, R \leq U$, $U(I, I) \leq I$, $LU \leq UB^2$, $RU \leq UA^2$.
(2) $\alpha U(\varphi, \psi) = \sup\{\alpha\varphi, \alpha\psi\}$ for all α, φ, ψ.

Notice that all the inequalities in (1) apart from $L, R \leq U$ are in fact equalities.

Exercise 7.11. Let $\mathscr{S}$ be the IOS of example 4.8. Show that the element $U = \lambda s.\{L(s), R(s)\}$ satisfies condition (1) of exercise 7.10.

Hint. Show that whenever $\mathscr{H} \subseteq \mathscr{F}$ and $\varphi = \cup \mathscr{H}$, then $\psi\varphi\chi = \sup\{\psi\theta\chi / \theta \in \mathscr{H}\}$ for all ψ, χ. Therefore, condition (2) of exercise 7.10 is satisfied.

Exercise 7.12*.** Let U satisfy condition (1) of exercise 7.10. Construct an element $\bar{\omega}$ recursive in U such that $\alpha\bar{\omega}\varphi = \sup_n \alpha\bar{n}\varphi$ for all α, φ.

Hint. Take $\bar{\omega}$ as in exercise 7.7 or, as suggested by N. Georgieva, $\bar{\omega} = \bar{1}[U(L, R^2)]$.

While O is the greatest (and unique) lower bound of L, R in the examples considered so far, we shall see in chapters 25, 30 that sometimes these elements have a greatest lower bound V satisfying the additional assumptions considered below.

Exercise 7.13. Show that the following are equivalent.

(1) $V \leq L, R$, $I \leq V(I, I)$.
(2) $V(\varphi, \psi) = \inf\{\varphi, \psi\}$ for all φ, ψ.

The element O does not satisfy (1) since $O(I, I) < I$.

Exercise 7.14. Show that the following are equivalent.

(1) $V \leq L, R$, $I \leq V(I, I)$, $VB^2 \leq LV$, $VA^2 \leq RV$.
(2) $\alpha V(\varphi, \psi) = \inf\{\alpha\varphi, \alpha\psi\}$ for all α, φ, ψ.

As in the case of U, all the inequalities in (1) except $V \leq L, R$ are in fact equalities. However, there is no analogue to exercises 7.9***, 7.12***, so one can only construct greatest lower bounds of finite subsets of $\mathscr{F}$.

CHAPTER 8

Representation theorems

Three theorems on the representability of primitive recursive and partial recursive number theoretic functions are established in this chapter, providing a way of incorporating Ordinary Recursion Theory within the general axiomatic theory.

Let $f:\omega^n \dashrightarrow \omega$, $n>0$. An element φ *weakly represents* f iff $\overline{s_1}\ldots\overline{s_n}\varphi = \overline{f(s_1,\ldots,s_n)}$ whenever $f(s_1,\ldots,s_n)\downarrow$. An element φ *represents* f iff φ weakly represents f and whenever $f(s_1,\ldots,s_n)\uparrow$, then $\overline{s_1}\ldots\overline{s_n}\varphi = O$. We write simply $\overline{s_1}\ldots\overline{s_n}\varphi = \overline{f(s_1,\ldots,s_n)}$, meaning $\overline{f(s_1,\ldots,s_n)} = O$ whenever $f(s_1,\ldots,s_n)\uparrow$.

Notice that whenever f is an extension of g and φ weakly represents f, then φ weakly represents g. Obviously, whenever f is a total function, then φ weakly represents f iff φ represents f.

The following statement is called the Representation Theorem for primitive recursive functions.

Proposition 8.1. Let Ψ be a set of partial number theoretic functions; let $\mathscr{B} \subseteq \mathscr{F}$ and suppose that all the functions in Ψ are representable by members of $\mathscr{B}$. Let f be a function primitive recursive in Ψ. Then there is an element φ primitive recursive in $\mathscr{B}$ such that φ represents f. In particular, all primitive recursive functions are representable by primitive recursive elements.

Proof. By induction on the construction of f.

1. If $f=\lambda s.0$, then take $\varphi = \langle\bar{0}\rangle H$. It follows that $\bar{s}\varphi = \bar{0}\bar{s}H = \bar{0}\langle I\rangle = \bar{0}$ for all s.

2. If $f=\lambda s.s+1$, then take $\varphi = R$. It follows that $\bar{s}\varphi = \bar{s}R = \overline{s+1}$ for all s.

3. If $f=\lambda s_1\ldots s_n.s_i$, $1\le i\le n$, then take $\varphi = H^{n-i}G_{i-1}H^{i-1}$. It follows that

$$\overline{s_1}\ldots\overline{s_n}\varphi = \overline{s_1}\ldots\overline{s_i}G_{i-1}H^{i-1} = \overline{s_i}\overline{s_1}\ldots\overline{s_{i-1}}H^{i-1} = \overline{s_i}.$$

4. If $f\in\Psi$, then f is represented by a member of $\mathscr{B}$.

5. Let h be m-ary, $g_1,\ldots,g_m$ n-ary, their representing elements $\chi,\psi_1,\ldots,\psi_m$ and

$$f=\lambda s_1\ldots s_n.h(g_1(s_1,\ldots,s_n),\ldots,g_m(s_1,\ldots,s_n)).$$

Then take $\varphi = D_n\langle\psi_1\rangle_n\ldots D_n\langle\psi_{m-1}\rangle_n\psi_m\chi$. The element φ is primitive recursive in $\chi,\psi_1,\ldots,\psi_m$ and

$$\overline{s_1}\ldots\overline{s_n}\varphi = \overline{s_1}\ldots\overline{s_n}\psi_1\ldots\overline{s_1}\ldots\overline{s_n}\psi_m\varphi,$$

hence φ represents f.

6. Let h be unary, g be ternary, their representing elements χ, ψ and

$$f(s,0) = h(s), \quad f(s,t+1) = g(s,t,f(s,t)).$$

Taking $\varphi = D^2\langle R\Delta(\langle I\rangle, (\langle I\rangle, D_2))\rangle\Delta(\chi,\psi)$, we get

$$\begin{aligned}\bar{s}\bar{0}\varphi &= \bar{s}\bar{0}\langle I\rangle(\langle I\rangle, D_2)\chi = \bar{s}\chi, \quad \bar{s}\overline{t+1}\varphi = \bar{s}\overline{t+1}(\langle I\rangle, D_2)^{t+2}\chi\psi^{t+1}\\ &= \bar{s}\bar{t}\bar{s}\bar{t}(\langle I\rangle, D_2)^{t+1}\chi\psi^t\psi = \bar{s}\bar{t}\bar{s}\bar{t}\varphi\psi.\end{aligned}$$

The element φ is primitive recursive in χ, ψ and an easy induction on t shows that φ represents f. This completes the proof.

Throughout proposition 8.1 and its proof 'weakly represent' can be substituted for 'represent'.

Two Representation Theorems for partial recursive functions follow.

Proposition 8.2. Let Ψ be a set of partial number theoretic functions, $\mathscr{B} \subseteq \mathscr{F}$ and suppose that all functions in Ψ are weakly representable by members of $\mathscr{B}$. Let f be a function μ-recursive in Ψ. Then there is an element φ recursive in $\mathscr{B}$ which weakly represents f. In particular, all the *general recursive functions* (i.e., total partial recursive functions) are representable by recursive elements.

Proof. We complete the proof of 8.1 (weak representability version) by adding one more case.

7. Let g be $n+1$-ary, let ψ weakly represent g and

$$f = \lambda s_1 \ldots s_n . \mu t(g(s_1, \ldots, s_n, t) = 0).$$

Taking $\psi_1 = \psi(L, [I]R)$, $\rho = G_n H^n$ (ρ represents $\lambda s_1 \ldots s_{n+1} . s_{n+1}$), $\psi_2 = D_{n+1} \circ \langle\psi_1\rangle_{n+1} C_{n+1}(\rho L^2, R^2)$ and $\sigma = R[\psi_2]$, we get

$$\begin{aligned}\bar{s}_1 \ldots \bar{s}_n \bar{t}\psi_2 &= \bar{s}_1 \ldots \bar{s}_n \bar{t}\psi_1(\bar{s}_1 \ldots \bar{s}_n \bar{t}L, \bar{s}_1 \ldots \bar{s}_n \bar{t}R)(\rho L^2, R^2)\\ &= \bar{s}_1 \ldots \bar{s}_n \bar{t}\psi_1(\bar{t}L^2, \bar{s}_1 \ldots \bar{s}_n \overline{t+2});\end{aligned}$$

hence

$$\bar{s}_1 \ldots \bar{s}_n \bar{t}\sigma = \bar{s}_1 \ldots \bar{s}_n \bar{t}\psi_1(\bar{t}L, \bar{s}_1 \ldots \bar{s}_n \overline{t+1}\sigma).$$

Take $\varphi = \bar{1}[\sigma]$; this element is recursive in ψ. Suppose that $f(s_1, \ldots, s_n) = t$. Then $g(s_1, \ldots, s_n, r)\downarrow$ for all $r \leq t$, $g(s_1, \ldots, s_n, r) > 0$ for all $r < t$, while $g(s_1, \ldots, s_n, t) = 0$. Therefore, $\bar{s}_1 \ldots \bar{s}_n \bar{t}\psi_1 = L$ and $\bar{s}_1 \ldots \bar{s}_n \bar{r}\psi_1 = R$ for $r < t$; hence

$$\bar{s}_1 \ldots \bar{s}_n \bar{r}\sigma = \bar{s}_1 \ldots \bar{s}_n \overline{r+1}\,\sigma$$

for $r < t$. Finally, it follows that

$$\begin{aligned}\bar{s}_1 \ldots \bar{s}_n \varphi &= \bar{s}_1 \ldots \bar{s}_n \bar{0}\sigma[\sigma] = \bar{s}_1 \ldots \bar{s}_n \bar{t}\sigma[\sigma] = \bar{s}_1 \ldots \bar{s}_n \bar{t}\psi_1(\bar{t}L, \bar{s}_1 \ldots \bar{s}_n \overline{t+1}\sigma)[\sigma]\\ &= \bar{t}L[\sigma] = \bar{t},\end{aligned}$$

which completes the proof.

Proposition 8.3*.** Let Ψ be a set of partial number theoretic functions, let $\mathscr{B} \subseteq \mathscr{F}$ and suppose that all functions in Ψ are representable by members of $\mathscr{B}$. Let f be a function μ-recursive in Ψ. Then there is an element φ recursive in $\mathscr{B}$ which represents f. In particular, all partial recursive functions are representable by recursive elements.

Proof. Using the proofs of 8.1, 8.2, it suffices to complete point 7.

Let g, f, ψ_1, ψ_2, σ, φ be the same as in the previous proof and $f(s_1\ldots,s_n)\uparrow$. We have to show that $\bar{s_1}\ldots\bar{s_n}\varphi=O$. There are two possibilities.

a. There is a t such that $g(s_1,\ldots,s_n,t)\uparrow$, while $g(s_1,\ldots,s_n,r)>0$ for $r<t$. It follows that $\bar{s_1}\ldots\bar{s_n}\bar{t}\psi_1=O$; hence

$$\bar{s_1}\ldots\bar{s_n}\varphi=\bar{s_1}\ldots\bar{s_n}\bar{t}\psi_1(\bar{t}L,\bar{s_1}\ \ldots\overline{s_n t+1}\sigma)[\sigma]=O.$$

b. $g(s_1,\ldots,s_n,r)>0$ for all r. Therefore, $\bar{s_1}\ldots\bar{s_n}\bar{r}\psi_1=R$ for all r; hence

$$\bar{s_1}\ldots\bar{s_n}\bar{r}\psi_2=\bar{s_1}\ldots\bar{s_n}\bar{r}\psi_1(\bar{r}L^2,\bar{s_1}\ldots\overline{s_n r+2})=\bar{s_1}\ldots\overline{s_n r+1}R$$

for all r, which implies $\bar{s_1}\ldots\bar{s_1}\bar{0}R[\psi_2]=O$ by (£££). We get

$$\bar{s_1}\ldots\bar{s_n}\varphi=\bar{s_1}\ldots\bar{s_n}\bar{1}[\sigma]=\bar{s_1}\ldots\bar{s_n}\bar{0}\sigma[\sigma]=O[\sigma]=O,$$

which completes the proof.

Propositions 8.1, 8.3*** are reversed in the exercises to ensure that whenever a number theoretic function is representable by a (primitive) recursive element, then it is partial (respectively primitive) recursive.

While it is shown in the exercises below that the functions representable by prime recursive elements are quite trivial, all partial recursive functions are representable by elements prime recursive in several initial ones, such as $C,P,Q,\langle L\rangle,\langle R\rangle$. (The latter assertion follows by 7.11, 8.3***.) Moreover, an examination of the proofs of 8.1–8.3*** shows that the resources of IOS are only partially employed. The properties of $\Pi,[\quad]$ used are $\bar{0}(\varphi,\psi)=\varphi$, $\overline{n+1}(\varphi,\psi)=\bar{n}\psi$, $\bar{0}[\varphi]=I$ and $\overline{n+1}[\varphi]=\bar{n}\varphi[\varphi]$ together with (£££) in case. Instead of $\langle\varphi\rangle_n$ it suffices to have an element φ_n such that $\bar{s_1}\ldots\bar{s_n}\varphi_n=\varphi\bar{s_1}\ldots\bar{s_n}$ for all $s_1,\ldots,s_n$. (If $n>1$, then $\varphi_n=P^{n-1}\varphi_1Q^{n-1}$ would do.) Therefore, representability of partial recursive functions could be established in certain simpler algebraic systems. However, we shall not pursue further representability results since the theorems given above are sufficient for our present purposes.

EXERCISES TO CHAPTER 8

Exercise 8.1. Let $J:\omega^2\to\omega$ be injective and partial recursive. Show that there are recursive elements ρ,σ which correspond to J in the sense of exercise 6.7**, i.e. $\bar{s}\bar{t}\rho=\overline{J(s,t)}$ and $\overline{J(s,t)}\sigma=\bar{s}\bar{t}$ for all s,t.

Hint. The existence of ρ follows by 8.2. Let $f_i=\lambda s.\psi_i(\mu t(J(\psi_0(t),\psi_1(t))=s))$, $i=0,1$, where ψ_0, ψ_1 are the inverses of $\lambda st.2^s(2t+1)-1$ considered in chapter 2. Take σ_0,σ_1 to weakly represent f_0,f_1 by 8.2, then take $\sigma=D\langle\sigma_0\rangle\sigma_1$.

It should be mentioned that the existence of elements ρ,σ which correspond to the Cantor's pairing function was assumed as an axiom in an earlier version of IOS. However, D. Skordev proved that the existence of such elements follows from axiom μA_3.

Exercise 8.2***. Let U satisfy condition (1) of exercise 7.8, Ψ be a set of partial number theoretic functions representable by members of $\mathscr{B} \subseteq \mathscr{F}$ and f be partial recursive in Ψ. Show that f is representable by an element recursive in $\{U\} \cup \mathscr{B}$.

Remark. By definition a n-ary function f is *partial recursive in* Ψ iff there is a $n+2$-ary function g primitive recursive in Ψ such that $f(s_1,\ldots,s_n) = t$ iff $\exists m(g(m,t,s_1,\ldots,s_n) = 0)$. If Ψ consists of total functions, then f is partial recursive in Ψ iff f is μ-recursive in Ψ. In particular, f is partial recursive iff f is μ-recursive, which enabled us to introduce the partial recursive functions via Kleene's definition given at the beginning of chapter 2. However, in general the notion of relative partial recursiveness is broader than the notion of relative μ-recursiveness. (Cf. Myhill [1961], Skordev [1963].)

Hint. Let g be a $n+2$-ary function primitive recursive in Ψ and $f(s_1,\ldots,s_n) = t$ iff $\exists m(g(m,t,s_1,\ldots,s_n) = 0)$. Take ψ to represent g by 8.1, a recursive element σ such that $\{\bar{k}\sigma/k \in \omega\} = \{\bar{t}\bar{m}/t, m \in \omega\}$ by exercise 8.1, and the element $\bar{\omega}$ of exercise 7.9***. Show that $\langle \bar{\omega}\sigma\langle D\rangle G\rangle_n \psi[R]$ represents f.

The next two exercises show that the unary functions representable by prime recursive elements are exactly those satisfying one of the following conditions.

(1) $\exists l \forall n(g(l+n) = g(l) + n)$.
(2) $\exists l \exists k > 0 \forall n \geq l(g(n+k) = g(n))$,
i.e. $\exists l \exists k > 0 \forall n(g(l+n) = g(l + \mathrm{rem}(n,k)))$.

Notice that while all such functions are primitive recursive in $\lambda s.\uparrow$, there are primitive recursive functions which satisfy neither (1) nor (2), e.g. $\lambda s.2s$.

Exercise 8.3. Show that if g satisfies (1) or (2), then it is representable by a prime recursive element ψ.

Hint. Let $g(l+n) = g(l) + n$ for all n. If $g(l)\downarrow$, then take $\psi = (\overline{g(0)},\ldots,\overline{g(l-1)}, R^{g(l)})$; if $g(l)\uparrow$, then take $\psi = (\overline{g(0)},\ldots,\overline{g(l-1)}, O)$.

Let $g(l+n) = g(l+\mathrm{rem}(n,k))$ for a certain $k > 0$ and all n. Then take $\psi = (\overline{g(0)},\ldots,\overline{g(l-1)}, R[(\overline{g(l)}L,\ldots,\overline{g(l+k-1)}L, R)])$.

Exercise 8.4***. Prove that whenever a prime recursive element ψ represents $g:\omega \longrightarrow \omega$, then g satisfies either (1) or (2).

Hint. We sketch a proof based on the unwinding method. There is a polynomial element φ such that $\psi = \bar{I}[\varphi]$. Let $m, \varphi_0,\ldots,\varphi_m$ correspond to φ by 4.12. If $\{l/\bar{I}_l\psi \neq O\}$ is finite, then g satisfies both (1) and (2). Otherwise, take $m+2$ distinct members $l_0,\ldots,l_{m+1}$ of this set. For each $i \leq m+1$ start constructing a sequence $\beta_{i,0}, \beta_{i,1},\ldots$ such that $\beta_{i,0} = R^{l_i}L$ and $R^{l_i}\psi = \beta_{i,n}\varphi[\varphi]$. By proposition 4.12, either $\beta_{i,n}\varphi \in \mathscr{D}$ or $\beta_{i,n}\varphi = \varphi_{m_i}$ for a certain $m_i \leq m$. Supposing $\beta_{i,n}\varphi = I$, one gets $\bar{I}_i\psi = I$ contrary to the fact that ψ represents g. If $\beta_{i,n}\varphi = \beta R$, then take $\beta_{i,n+1} = \beta$. If $\beta_{i,n}\varphi = \beta L$ for certain i, n, then stop constructing all the sequences. If $\beta_{i,n}\varphi = \varphi_{m_i}$, then stop constructing the i-th one.

No sequence could be infinite since $\forall n(\beta_{i,n}\varphi = \beta_{i,n+1}R)$ would imply $\overline{l_i}\psi = O$ by (£££) contrary to the choice of l_i.

If $\beta_{i,n}\varphi = \beta L$ for certain i, n, then $R^{l_i}\psi = \beta$, hence $\overline{l_i}\psi = L\beta$. Therefore, $\beta = R^{g(l_i)}$ which implies $\overline{l_i + n}\,\psi = \bar{n}R^{g(l_i)} = \overline{g(l_i) + n}$ for all n, i.e. g satisfies (1).

Finally, one may get $m_0, \dots, m_{m+1} \le m$ such that $R^{l_i}\psi = \varphi_{m_i}[\varphi]$ for all $i \le m+1$, in which case $m_i = m_j$ for certain $i \ne j \le m+1$, hence $R^{l_i}\psi = R^{l_j}\psi$ and g satisfies (2). The proof is complete.

Similarly, the n-ary functions representable by prime recursive elements can be shown to compose the class Ψ introduced inductively as follows.

1. If g is unary and satisfies (1) or (2), then $g \in \Psi$.
2. If g is $n+1$-ary, $n > 0$, there are $k > 0$ and l such that $\lambda s_1 \dots s_n . g(s_1, \dots, s_n, i) \in \Psi$, $i \le l + k - 1$, and $g(s_1, \dots, s_n, s + k) = g(s_1, \dots, s_n, s)$ for all $s \ge l$, then $g \in \Psi$.

Of course, all such functions are primitive recursive in $\lambda s.\uparrow$.

8.4***

Exercise 8.5*.** Prove that whenever $f : \omega^n \dashrightarrow \omega$ is represented by a recursive element φ, then f is partial recursive. (If necessary, make use of the Normal Form Theorem 9.3 established in the next chapter.)

Hint. The unwinding method helps once again. Take a primitive element ψ such that $\varphi = \bar{1}[\psi]$ by 9.3. Fix a construction of ψ and for all α introduce a set $\mathrm{Sub}(\alpha\psi) \subseteq \mathscr{D}$ ('the sub elements of $\alpha\psi$') as follows.

$$
\begin{aligned}
\mathrm{Sub}(\alpha\beta) &= \{\alpha\beta\},\\
\mathrm{Sub}(\alpha\psi_1\psi_2) &= \cup\{\mathrm{Sub}(\beta\psi_2)/\beta \in \mathrm{Sub}(\alpha\psi_1)\},\\
\mathrm{Sub}(I(\psi_1, \psi_2)) &= \mathrm{Sub}(\psi_1) \cup \mathrm{Sub}(\psi_2),\\
\mathrm{Sub}(\alpha L(\psi_1, \psi_2)) &= \mathrm{Sub}(\alpha\psi_1),\\
\mathrm{Sub}(\alpha R(\psi_1, \psi_2)) &= \mathrm{Sub}(\alpha\psi_2),\\
\mathrm{Sub}(I\langle\psi_1\rangle) &= \{\beta\bar{n}/\beta \in \mathrm{Sub}(\psi_1) \,\&\, n \in \omega\},\\
\mathrm{Sub}(\alpha L\langle\psi_1\rangle) &= \{\beta L/\beta \in \mathrm{Sub}(\alpha\psi_1)\},\\
\mathrm{Sub}(\alpha R\langle\psi_1\rangle) &= \{\beta R/\beta \in \mathrm{Sub}(\alpha\langle\psi_1\rangle)\}
\end{aligned}
$$

Notice that for all α and $\beta \in \mathrm{Sub}(\alpha\psi)$ there is a $\gamma \in \mathscr{D}$ such that $\gamma\alpha\psi = \beta$. Moreover,

$$\forall \beta \in \mathrm{Sub}(\alpha\psi)(\beta\rho \le \beta\sigma) \Rightarrow \alpha\psi\rho \le \alpha\psi\sigma.$$

Given $s_1, \dots, s_n$, take $\alpha_0 = \bar{s}_1 \dots \bar{s}_n \bar{1}$, $\mathscr{A}_0 = \{\alpha_0\}$, $\mathscr{A}_{i+1} = \cup\{\mathrm{Sub}(\alpha_i'\psi)/\alpha_i' R \in \mathscr{A}_i\}$ and $\mathscr{A} = \cup_i \mathscr{A}_i$. An easy induction on i shows that $\forall \alpha \in \mathscr{A} \exists \gamma(\gamma\alpha_0[\psi] = \alpha[\psi])$, which in particular gives $I \notin \mathscr{A}$ since $\gamma\alpha_0[\psi] = [\psi]$ would imply the false equality $L\gamma\bar{s}_1 \dots \bar{s}_n\varphi = I$.

In order to compute $f(s_1, \dots, s_n)$ start generating the members of $\mathscr{A}$ consecutively. There are two possibilities.

First, one may get $\alpha = \alpha' L$ for a certain $\alpha \in \mathscr{A}$. There is a γ such that $\gamma\alpha_0[\psi] = \alpha[\psi] = \alpha'$; hence $\bar{s}_1 \dots \bar{s}_n\varphi \ne O$. Therefore, $f(s_1, \dots, s_n)\downarrow$ and $t = f(s_1, \dots, s_n)$ is effectively derived out of α since $\gamma\bar{t} = \alpha'$. The computation terminates.

Secondly, the process may fail to terminate if $\alpha=\alpha' R$ for all $\alpha\in\mathscr{A}$, in which case one expects that $f(s_1,...,s_n)\uparrow$. Actually, consider the regular segment $\mathscr{E}=\{\theta/\forall\alpha\in\mathscr{A}(\alpha\theta L\leq\alpha[\psi])\}$. Let $\theta\in\mathscr{E}$ and $\alpha\in\mathscr{A}$. Then $\alpha(I,\psi\theta)L=\alpha' R(I,\psi\theta)L=\alpha'\psi\theta L$. It follows that $\alpha'\psi\theta L\leq\alpha'\psi[\psi]$ since $\beta\theta L\leq\beta[\psi]$ for all $\beta\in\mathrm{Sub}(\alpha'\psi)$. Therefore, $\alpha(I,\psi\theta)L\leq\alpha[\psi]$; hence $[\psi]\in\mathscr{E}$ by μA_3. In particular, $\alpha_0[\psi]L\leq\alpha_0[\psi]$ implies $\overline{s_1}...\overline{s_n}\varphi\notin\mathscr{D}$, i.e. $f(s_1,...,s_n)\uparrow$. This completes the proof.

The corresponding result for primitive recursiveness is to be found in the exercises to chapter 22.

CHAPTER 9

Basic theorems of recursion theory

Some standard theorems characteristic of Recursion Theory are established in this chapter: a Normal Form Theorem, First and Second Recursion Theorems, an Enumeration Theorem.

We begin with several normal form results throwing light on the structure of the recursive elements and mappings. In order to give more precise formulations we introduce the following special kinds of elements polynomial or primitive in a subset $\mathscr{B}$ of $\mathscr{F}$.

An element φ is *strictly polynomial in* $\mathscr{B}$ iff

$$\varphi \in cl(\{L, R\} \cup \mathscr{B} / \lambda\theta.\theta L, \lambda\theta.\theta R, \Pi),$$

which by the distributive law is equivalent to

$$\varphi \in cl(cl(\{L, R\} \cup \mathscr{B} / \lambda\theta.\theta L, \lambda\theta.\theta R)/\Pi).$$

Of course, all such elements are polynomial in $\mathscr{B}$, while 4.1, 4.12 imply that all polynomial elements are strictly polynomial (in $\varnothing$).

Let a finite set $\mathscr{B}_0$ of primitive elements be fixed as in chapter 7 such that $cl(\mathscr{B}_0/\circ, [\ \ ])$ contains all recursive elements. An element φ is *strictly primitive in* $\mathscr{B}$ iff φ is strictly polynomial in $\mathscr{B}_0 \cup \langle \mathscr{B} \rangle$. Notice that whenever φ is strictly primitive in $\mathscr{B}$, then so are φL, φR. All polynomial elements are strictly primitive (in $\varnothing$). It is also immediate that all the elements strictly primitive in $\mathscr{B}$ are primitive in $\mathscr{B}$. However, it can not be claimed that all the elements polynomial (primitive) in $\mathscr{B}$ are strictly polynomial (respectively strictly primitive) in $\mathscr{B}$ for all $\mathscr{B}$; relevant counterexamples will be given in the exercises to chapter 21.

Proposition 9.1. Let $\mathscr{C} \subseteq \mathscr{F}$ and $\varphi \in cl(\mathscr{C}/\circ, [\ \ ])$. Then there is an element σ strictly polynomial in $\mathscr{C}$ such that $\varphi = \bar{1}[\sigma]$.

Proof. We first prove by induction on the construction of φ that $\varphi = \rho[\sigma]$ with certain ρ, σ strictly polynomial in $\mathscr{C}$.

If $\varphi \in \mathscr{C}$, then $\varphi = R[\varphi L]$ by 6.8.

Let $\varphi = \rho[\sigma]$ and $\psi = \rho_1[\sigma_1]$. Then

$$\varphi\psi = \rho\bar{1}[((\rho_1 R^2, \sigma\bar{1}), \bar{0}, \sigma_1 R^2)]$$

by 6.15, while

$$[\varphi] = \bar{1}[((\bar{0}, \rho R^2), \bar{1}, \sigma R^2)]$$

by 6.16.

Now let $\varphi = \rho[\sigma]$ with ρ, σ strictly polynomial in $\mathscr{C}$. Then $\varphi = \bar{1}[(\rho R^2, \bar{0}, \sigma R^2)]$ by 6.14, which completes the proof.

The following is a slightly modified version of the Normal Form Theorem for elements prime recursive in $\mathscr{B}$ established in Georgieva [1980] and already used in the hints to exercises 6.14***, 8.4***.

Proposition 9.2. If φ is prime recursive in $\mathscr{B}$, then $\varphi = \bar{1}[\sigma]$ with a certain σ strictly polynomial in $\mathscr{B}$.

This follows from 7.5 and 9.1, taking $\mathscr{C} = \{L, A\} \cup \mathscr{B}$.

The following assertion is called the Normal Form Theorem for elements recursive in $\mathscr{B}$.

Proposition 9.3. If φ is recursive in $\mathscr{B}$, then $\varphi = \bar{1}[\sigma]$ for some σ strictly primitive in $\mathscr{B}$.

This follows from 9.1, taking $\mathscr{C} = \mathscr{B}_0 \cup \langle \mathscr{B} \rangle$.

We observe a curious phenomenon, namely, a strict stratification of the initial operations of an IOS. They are separately ordered as follows: translation, multiplication (right), pairing operation, iteration, multiplication (left). For, the Normal Form Theorem shows that the elements recursive in $\mathscr{B}$ can be constructed in a few stages, with different operations used in each of them. In the first stage members of $\mathscr{B}$ are taken and $\langle \quad \rangle$ is applied to each of them once. Secondly, elements from $\mathscr{B}_0$ are allowed and the operations $\lambda\theta.\theta L, \lambda\theta.\theta R$ are applied repeatedly. At the third stage Π is applied repeatedly. After that $[\quad]$ is applied just once. Finally, the elements so obtained are multiplied to the left by LR.

Certain modifications of 9.2, 9.3 can be obtained by applying exercise 4.9 to the element σ, stratifying the initial IOS-operations in another way.

Analogous normal form theorems in Ordinary Recursion Theory and the theory of Skordev combinatory spaces show that it is sufficient to use the least number operator, respectively the operation iteration just once.

The following Normal Form Theorem for mappings recursive in $\mathscr{B}$ is an analogue to that of Skordev [1978].

Proposition 9.4. If Γ is a n-ary mapping recursive in $\mathscr{B}$, then

$$\Gamma = \lambda\theta_1 \ldots \theta_n . \bar{1}[\varphi(I, \langle\theta_1\rangle, \ldots, \langle\theta_n\rangle)]$$

for a certain φ strictly primitive in $\mathscr{B}$.

Proof. Throughout this proof ρ will stand for $(I, \langle\theta_1\rangle, \ldots, \langle\theta_n\rangle)$. First we prove by induction on the construction of Γ that $\Gamma = \lambda\theta_1 \ldots \theta_n . \alpha[\psi\rho\tau]$ for certain ψ, τ recursive in $\mathscr{B}$.

Let $\Gamma = \lambda\theta_1 \ldots \theta_n . \langle\theta_i\rangle$, $1 \leq i \leq n$. If $i < n$, then $\Gamma = \lambda\theta_1 \ldots \theta_n . R[\bar{i}\rho L]$. If $i = n$, then $\Gamma = \lambda\theta_1 \ldots \theta_n . R[R^n\rho L]$.

Let $\Gamma = \lambda\theta_1\ldots\theta_n.\psi$, $\psi \in \mathscr{B}_0 \cup \langle\mathscr{B}\rangle$. Then $\Gamma = \lambda\theta_1\ldots\theta_n.R[\psi L^2\rho]$.

Suppose that $\Gamma_1 = \lambda\theta_1\ldots\theta_n.\alpha_1[\psi_1\rho\tau_1]$ and $\Gamma_2 = \lambda\theta_1\ldots\theta_n.\alpha_2 L[\psi_2\rho\tau_2]$.

If $\Gamma = \lambda\theta_1\ldots\theta_n.\Gamma_1(\theta_1,\ldots,\theta_n)\Gamma_2(\theta_1,\ldots,\theta_n)$, then 6.15 gives $\Gamma = \lambda\theta_1\ldots\theta_n.\alpha_1\bar{1}[\sigma]$, where

$$\begin{aligned}\sigma &= ((\alpha_2 R^2, \psi_1\rho\tau_1\bar{1}), \bar{0}, \psi_2\rho\tau_2 R^2) = \psi_3(I, \langle\theta_1\rangle\bar{0}, \langle\theta_1\rangle\bar{1}, \ldots, \langle\theta_n\rangle\bar{0}, \langle\theta_n\rangle\bar{1})\tau_3\\ &= \psi_3(I, \bar{0}\langle\theta_1\rangle_2, \bar{1}\langle\theta_1\rangle_2, \ldots, \bar{0}\langle\theta_n\rangle_2, \bar{1}\langle\theta_n\rangle_2)\tau_3\\ &= \psi_4(I, \langle\theta_1\rangle_2, \langle\theta_2\rangle_2, \ldots, \langle\theta_n\rangle_2)\tau_3\\ &= \psi_4(L, PR\langle\theta_1\rangle, \ldots, PR\langle\theta_n\rangle)(I, Q)\tau_3 = \psi\rho\tau\end{aligned}$$

with certain $\psi_3, \tau_3, \psi_4, \psi, \tau$ recursive in $\psi_1, \tau_1, \psi_2, \tau_2$.

If $\Gamma = \lambda\theta_1\ldots\theta_n.[\Gamma_1(\theta_1,\ldots,\theta_n)]$, then 6.16 gives $\Gamma = \lambda\theta_1\ldots\theta_n.\bar{1}[\sigma]$, where

$$\sigma = ((\bar{0}, \alpha_1 R^2), \bar{1}, \psi_1\rho\tau_1 R^2) = \psi\rho\tau$$

with certain ψ, τ recursive in ψ_1, τ_1.

Now let $\Gamma = \lambda\theta_1\ldots\theta_n.\alpha[\psi\rho\tau]$ with ψ, τ recursive in $\mathscr{B}$. There is by 9.3 an element φ_1 strictly primitive in $\mathscr{B}$ such that $\psi = \bar{1}[\varphi_1]$; hence

$$\psi\sigma\tau = \bar{1}[\varphi_1]\rho\tau = \bar{2}[(\rho\tau L, \varphi_1 R)]$$

by 6.13. It follows from 9.3 and exercise 6.1d that $\tau L = R^2[\varphi_2]$ with a certain φ_2 strictly primitive in $\mathscr{B}$. Therefore,

$$\psi\rho\tau = \bar{2}[(\rho R^2, \varphi_1 RL)[\varphi_2]] = \bar{2}\bar{1}[\sigma]$$

by 6.16, where

$$\sigma = ((\bar{0}, \rho R^4, \varphi_1 R\bar{2}), \bar{1}, \varphi_2 R^2) = ((\varphi_3, \rho R^4, \varphi_4), \varphi_5)$$

with $\varphi_3, \varphi_4, \varphi_5$ strictly primitive in $\mathscr{B}$. It follows from 6.16 that $\Gamma = \lambda\theta_1\ldots\theta_n.\alpha\bar{1}[\sigma_1]$, where

$$\sigma_1 = ((\bar{0}, \overline{23}), \bar{1}, \sigma R^2) = (\varphi_6, \varphi_7, (\varphi_8, \rho R^6, \varphi_9), \varphi_{10})$$

with $\varphi_6, \ldots, \varphi_{10}$ strictly primitive in $\mathscr{B}$. Finally, we get $\Gamma = \lambda\theta_1\ldots\theta_n.\bar{1}[\sigma_2]$ by 6.14, where

$$\sigma_2 = (\alpha\bar{3}, \bar{0}, \sigma_1 R^2) = \varphi\rho$$

with a certain φ strictly primitive in $\mathscr{B}$, using the equality $\langle\theta\rangle R = R\langle\theta\rangle$. This completes the proof.

In particular, one gets $\Gamma = \lambda\theta.\bar{1}[\varphi(I, \langle\theta\rangle)]$ whenever Γ is unary.

Three modified versions of the last normal form theorem follow.

Proposition 9.5. If Γ is a n-ary mapping recursive in $\mathscr{B}$, then

$$\Gamma = \lambda\theta_1\ldots\theta_n.\bar{1}[\varphi(I, \langle\psi_1\rangle, \ldots, \langle\psi_m\rangle, \langle\theta_1\rangle, \ldots, \langle\theta_n\rangle)]$$

with certain $\psi_1, \ldots, \psi_m \in \mathscr{B}$ and a strictly primitive φ.

Proof. The mapping Γ is recursive in a finite subset $\{\psi_1, \ldots, \psi_m\}$ of $\mathscr{B}$. Let Γ^* correspond to Γ by 7.19. Applying 9.4 to the recursive mapping Γ^*, we get the desired normal form of Γ.

Proposition 9.6. If Γ is a n-ary mapping recursive in $\mathscr{B}$, then

$$\Gamma = \lambda\theta_1 \ldots \theta_n . \bar{1}[\varphi(I, \langle(\theta_1, \ldots, \theta_n)\rangle)]$$

with φ strictly primitive in $\mathscr{B}$.

Proof. Let Γ^* correspond to Γ by 7.20. Then we get the desired normal form by applying 9.4 to Γ^*.

Proposition 9.7. If Γ is a n-ary mapping recursive in $\mathscr{B}$, then

$$\Gamma = \lambda\theta_1 \ldots \theta_n . \bar{1}[\varphi(I, \langle(\psi_1, \ldots, \psi_m, \theta_1, \ldots, \theta_n)\rangle)]$$

with $\psi_1, \ldots, \psi_m \in \mathscr{B}$ and a strictly primitive φ.

This follows from 7.19, 9.6.

The next statement is an improved version of the Normal Form Theorem for unary mappings prime recursive in $\mathscr{B}$ established in Georgieva [1980]. First, however, we make a definition. To each such mapping Γ assign a natural number $c(\Gamma)$ viz., the least n such that Γ has a construction in which the clause with $\lambda\theta.\theta$ occurs n times. Such a 'complexity' could also be introduced for unary mappings recursive, primitive recursive etc. in $\mathscr{B}$. Notice that in the case of recursiveness $c(\Gamma) \leq 1$ by 9.4.

Proposition 9.8. Let Γ be a unary mapping prime recursive in $\mathscr{B}$ and $c(\Gamma) = n$. Then

$$\Gamma = \lambda\theta . \bar{1}[\varphi(I, \theta\bar{4}, \ldots, \theta\overline{n+3})]$$

with a certain φ strictly polynomial in $\mathscr{B}$.

Proof. To begin with, we prove by induction on the construction of Γ (the clause $\lambda\theta.\theta$ assumed to appear n times) that

$$\Gamma = \lambda\theta . \alpha[\psi(I, \theta\alpha_1, \ldots, \theta\alpha_n)]$$

with a certain ψ polynomial in $\mathscr{B}$.

If $\Gamma = \lambda\theta.\theta$, then $\Gamma = \lambda\theta . R[R(I, \theta L)]$.

If $\Gamma = \lambda\theta.\psi$, $\psi \in \{L, A\} \cup \mathscr{B}$, then $\Gamma = \lambda\theta . R[\psi L]$.

Let $\Gamma_1 = \lambda\theta . \alpha[\psi_1(I, \theta\alpha_1, \ldots, \theta\alpha_k)]$, $\Gamma_2 = \lambda\theta . \beta[\psi_2(I, \theta\beta_1, \ldots, \theta\beta_l)]$ and $\Gamma = \lambda\theta . \Gamma_1(\theta)\Gamma_2(\theta)$.

Using 6.15, we obtain

$$\Gamma = \lambda\theta . \alpha'[\psi(I, \theta\alpha'_1, \ldots, \theta\alpha'_{k+l})]$$

with ψ polynomial in ψ_1, ψ_2.

The case of iteration is treated by means of 6.16.

Now let $\Gamma = \lambda\theta . \alpha[\psi(I, \theta\alpha_1, \ldots, \theta\alpha_n)]$, where ψ is polynomial in $\mathscr{B}$. It follows from 9.2 that $\psi = \bar{1}[\varphi_1]$ with φ_1 strictly polynomial in $\mathscr{B}$, hence

$$\psi(I, \theta\alpha_1, \ldots, \theta\alpha_n) = \bar{1}[\varphi_1](I, \theta\alpha_1, \ldots, \theta\alpha_n) = \bar{2}[((L, \theta\alpha_1 L, \ldots, \theta\alpha_n L), \varphi_1 R)]$$

by 6.13. We get $\Gamma = \lambda\theta . \alpha\bar{1}[\sigma]$ by 6.16, where

$$\sigma = ((\bar{0}, \bar{4}), \bar{1}, (\bar{2}, \theta\alpha_1\bar{2}, \ldots, \theta\alpha_n\bar{2}), \varphi_1 R^3) = (\varphi_2, \varphi_3, (\varphi_4, \theta\beta_1, \ldots, \theta\beta_n), \varphi_5)$$

with $\varphi_2,\ldots,\varphi_5$ strictly polynomial in $\mathscr{B}$. Let $\tau=(\beta_1,\ldots,\beta_n,I)$. Then

$$\sigma=(\varphi_2R^n,\varphi_3R^n,(\varphi_4R^n,\theta\overline{0},\ldots,\theta\overline{n-1}),\varphi_5R^n)\tau.$$

Substituting $R[\tau L]$ for τ and using exercise 6.1h, 6.14, we get the desired normal form. The proof is complete.

In particular, one gets $\Gamma=\lambda\theta.\overline{1}[\varphi(I,\theta\overline{4})]$ whenever $c(\Gamma)=1$. A normal form $\Gamma=\lambda\theta.\overline{1}[\varphi(I,\theta R^4)]$ can also be obtained in this case.

The assumption $c(\Gamma)=n$ above can obviously be replaced by $c(\Gamma)\leq n$. Moreover, 9.8 can be generalized to n-ary mappings, $n>1$.

It is also worth mentioning that the proof of 9.4 produces an element φ such that for all $\theta_1,\ldots,\theta_n$ the element $\varphi(I,\langle\theta_1\rangle,\ldots,\langle\theta_n\rangle)$ is uniformly strictly primitive in $\{\theta_1,\ldots,\theta_n\}\cup\mathscr{B}$, while in the case of 9.8 $\varphi(I,\theta\overline{4},\ldots,\theta\overline{n+3})$ is uniformly strictly polynomial in $\{\theta\}\cup\mathscr{B}$. The elements φ in 9.4–9.8 can be also modified by means of exercise 4.9.

While the natural concept of inductiveness was introduced at the beginning of chapter 5, the notion of recursiveness was taken as fundamental and has been regarded as central so far. We are now going to justify this by showing that the class of all inductive mappings is identical with that of all recursive ones. The following statement is the easy half of such a justification.

Proposition 9.9. If a mapping Γ is recursive, then it is inductive.

The proof is by induction on the construction of Γ, making use of the simple fact that composition of mappings preserves their inductiveness.

It should be mentioned that the desire to obtain 9.9 was the only reason for taking $\lambda\theta_1\ldots\theta_n.\psi$, $\psi\in\{I,L,R\}$ among the initial inductive mappings, for otherwise it would only follow that for every n-ary recursive mapping Γ there is a $n+3$-ary inductive mapping Γ_1 such that $\Gamma=\lambda\theta_1\ldots\theta_n.\Gamma_1(I,L,R,\theta_1,\ldots,\theta_n)$.

On the other hand, all inductive mappings turn out to be recursive. To ensure this one should prove that the μ-operation preserves recursiveness, i.e. whenever $\Gamma:\mathscr{F}^{n+1}\to\mathscr{F}$ is recursive, then so is $\lambda\theta_1\ldots\theta_n.\mu\theta.\Gamma(\theta_1,\ldots,\theta_n,\theta)$. In other words, one needs the First Recursion Theorem. In fact, propositions 6.38, 6.39 were the first steps toward such a theorem and we are now able to complete its proof in two final steps, using axiom μA_1 in the latter.

Proposition 9.10 (Third Recursion Lemma). The element $\mu\theta.\varphi[\psi(I,\theta\chi)]$ exists and is recursive in φ,ψ,χ for all φ,ψ,χ.

Proof. We show that the element in question equals $\varphi R^2\overline{1}[\sigma]$, where $\sigma=\Delta(\sigma_1,\sigma_2)$, $\sigma_1=(\chi\overline{1}\overline{1},R^2\overline{1},\overline{0},\psi(\overline{1}\overline{1},\varphi R^2\overline{2}))$, $\sigma_2=(\overline{0}\overline{1},R^2)$.

Using 6.13 and exercise 6.1i it follows that

$$\begin{aligned}\varphi[\psi(I,\varphi R^2\overline{1}[\sigma]\chi)]&=\varphi[\psi(\overline{0},\varphi R^2\overline{2})[(\chi\overline{0},\sigma R)]]=\varphi R^2\overline{1}[(\sigma_1,\sigma\sigma_2)]\\&=\varphi R^2\overline{1}[\sigma].\end{aligned}$$

Suppose that $\varphi[\psi(I, \tau\chi)] \leq \tau$. Take $\rho = [\psi(I, \tau\chi)]$, $\rho_1 = \Delta((\chi\rho, \rho, \rho), \chi\rho)$. Then

$$\sigma_1(I, \rho_1) = (\chi\rho, \rho, I, \psi(\rho, \varphi\rho\chi\rho)) = (\chi\rho, \rho, I, \psi(I, \varphi\rho\chi)\rho)$$
$$\leq (\chi\rho, \rho, I, \psi(I, \tau\chi)\rho) = (\chi\rho, \rho, \rho) = L\rho_1,$$
$$\sigma_2(I, \rho_1) = (\chi\rho, \rho_1\chi\rho) = (I, \rho_1)\chi\rho$$

and $\rho_1\chi\rho = R\rho_1$; hence $\sigma(I, \rho_1) \leq \rho_1$ by 6.30. Therefore, $R[\sigma] \leq \rho_1$ by 6.11, which implies

$$\varphi R^2\bar{1}[\sigma] \leq \varphi R^2\bar{0}\rho_1 = \varphi\rho \leq \tau.$$

This completes the proof.

Proposition 9.11. If Γ is a unary mapping prime recursive in $\mathscr{B}$ and $c(\Gamma) \leq 1$, then the element $\mu\theta.\Gamma(\theta)$ exists and is recursive in $\mathscr{B}$.

This follows from 9.8, 9.10.

Proposition 9.12* (Fourth Recursion Lemma). The element $\mu\theta.\varphi[\psi(I, \langle\theta\rangle)]$ is recursive in φ, ψ for all φ, ψ.

Proof. The element $\theta_0 = \mu\theta.\varphi[\psi(I, \langle\theta\rangle)]$ exists by 5.2*. It remains to show that it is recursive in φ, ψ.

Take $\varphi_1 = \langle\varphi\rangle C$, $\psi_1 = \langle\psi\rangle C(CL, PR)$ and $\chi_1 = QC$. The element $\theta_1 = \mu\theta.\varphi_1[\psi_1(I, \theta\chi_1)]$ is recursive in $\varphi_1, \psi_1, \chi_1$ by 9.10, hence it is recursive in φ, ψ. Using 6.37, 6.40, we get

$$\langle\varphi[\psi(I, \langle\theta\rangle)]\rangle = \langle\varphi\rangle C[\langle\psi\rangle C(\langle I\rangle, \langle\theta\rangle_2)C] = \varphi_1[\langle\psi\rangle C(C, P\langle\theta\rangle QC)]$$
$$= \varphi_1[\psi_1(I, \langle\theta\rangle\chi_1)]$$

for all θ. It follows in particular that

$$\langle\theta_0\rangle = \langle\varphi[\psi(I, \langle\theta_0\rangle)]\rangle = \varphi_1[\psi_1(I, \langle\theta_0\rangle\chi_1)];$$

hence $\theta_1 \leq \langle\theta_0\rangle$.

On the other hand, consider the simple segment

$$\mathscr{E} = \{\theta/\langle\theta\rangle \leq \theta_1\} = \{\theta/\langle\theta\rangle \leq \langle I\rangle\theta_1\}.$$

If $\theta \in \mathscr{E}$, then

$$\langle\varphi[\psi(I, \langle\theta\rangle)]\rangle = \varphi_1[\psi_1(I, \langle\theta\rangle\chi_1)] \leq \varphi_1[\psi_1(I, \theta_1\chi_1)] = \theta_1,$$

hence $\theta_0 \in \mathscr{E}$ by μA_1. Therefore, $\langle\theta_0\rangle = \theta_1$, hence $\theta_0 = L\theta_1(I, I)$, which completes the proof.

We call the technique used above the *translation method* since in order to solve the equality $\varphi[\psi(I, \langle\theta\rangle)] = \theta$ we have transformed ('translated') it by means of $\langle\ \ \rangle$ into the more tractable form $\varphi_1[\psi_1(I, \theta\chi_1)] = \theta$.

The following statement is called the First Recursion Theorem, being actually an analogue of theorem XXVI of Kleene [1952] and the theorem proved in Skordev [1979].

Proposition 9.13*. If Γ is a unary mapping recursive in $\mathscr{B}$, then the element $\mu\theta.\Gamma(\theta)$ is recursive in $\mathscr{B}$.

This follows from 9.12* since Γ has a normal form $\lambda\theta.\varphi[\psi(I,\langle\theta\rangle)]$ with φ,ψ recursive in $\mathscr{B}$ by 9.4. Next, we give a parametrized version of the First Recursion Theorem.

Proposition 9.14*. If Γ is a $n+1$-ary mapping recursive in $\mathscr{B}$, then the mapping $\Gamma_1=\lambda\theta_1\ldots\theta_n.\mu\theta.\Gamma(\theta_1,\ldots,\theta_n,\theta)$ is recursive in $\mathscr{B}$.

Proof. We recall that the existence of Γ_1 is ensured by 5.2*. The element θ_0 in the proof of 9.12* is constructed uniformly in φ,ψ; in essence, we have obtained a recursive binary mapping Γ_0 such that $\Gamma_0(\varphi,\psi)=\mu\theta.\varphi[\psi(I,\langle\theta\rangle)]$ for all φ,ψ. On the other hand, it follows immediately from 9.4 that

$$\Gamma=\lambda\theta_1\ldots\theta_{n+1}.\bar{1}[\Gamma_2(\theta_1,\ldots,\theta_n)(I,\langle\theta_{n+1}\rangle)]$$

with a certain mapping Γ_2 recursive in $\mathscr{B}$. Therefore,

$$\Gamma_1=\lambda\theta_1\ldots\theta_n.\Gamma_0(\bar{1},\Gamma_2(\theta_1,\ldots,\theta_n));$$

hence Γ_1 is recursive in $\mathscr{B}$ by 7.18.

Proposition 9.15*. If a mapping Γ is inductive, then it is recursive.

This follows from 9.14*.

The First Recursion Theorem makes it possible to solve systems of inequalities as well.

Proposition 9.16*. Let a system of inequalities

$$\Gamma_i(\theta_0,\ldots,\theta_n)\le\theta_i,\ i\le n, \tag{1}$$

be given, where $n>0$ and $\Gamma_0,\ldots,\Gamma_n$ are recursive in $\mathscr{B}$. Then there are elements $\varphi_0,\ldots,\varphi_n$ recursive in $\mathscr{B}$ such that $\varphi_0,\ldots,\varphi_n$ satisfy (1) and whenever $\tau_0,\ldots,\tau_n$ satisfy (1), then $\varphi_i\le\tau_i$ for all $i\le n$.

Proof. The system (1) may be reduced to a single inequality in a manner similar to that of Skordev [1979]. Namely, the mapping

$$\Gamma=\lambda\theta.(\Gamma_0(\bar{0}\theta,\ldots,\overline{n-1}\,\theta,R^n\theta)),\ldots,\Gamma_n(\bar{0}\theta,\ldots,\overline{n-1}\,\theta,R^n(\theta))$$

is recursive in $\mathscr{B}$, hence so is $\theta_0=\mu\theta.\Gamma(\theta)$ by 9.13*. Take $\varphi_i=\bar{i}\theta_0$ for $i<n$ universal element (mapping).

$$\Gamma_i(\varphi_0,\ldots,\varphi_n)=\bar{i}\Gamma(\theta_0)=\bar{i}\theta_0=\varphi_i,\ i<n,$$
$$\Gamma_n(\varphi_0,\ldots,\varphi_n)=R^n\Gamma(\theta_0)=R^n\theta_0=\varphi_n;$$

hence $\varphi_0,\ldots,\varphi_n$ satisfy (1).

Let $\tau_0,\ldots,\tau_n$ be another solution to (1). Then

$$\Gamma((\tau_0,\ldots,\tau_n))=(\Gamma_0(\tau_0,\ldots,\tau_n),\ldots,\Gamma_n(\tau_0,\ldots,\tau_n))\le(\tau_0,\ldots,\tau_n),$$

hence $\theta_0\le(\tau_0,\ldots,\tau_n)$, which implies $\varphi_i\le\tau_i$ for all $i\le n$. This completes the proof.

In reality, the mappings $\Gamma_0,\ldots,\Gamma_n$ in (1) may be of various arities $m\le n$ since any such mapping Γ_i may be replaced by the n-ary mapping $\lambda\theta_1\ldots\theta_n.\Gamma_i(\theta_1,\ldots,\theta_m)$.

The remainder of this chapter is devoted to universal elements and mappings.

Let $\mathscr{U} \subseteq \mathscr{F}$ and let $\mathscr{M}$ be a set of unary mappings over $\mathscr{F}$. An element σ is *universal for* $\mathscr{U}$ iff $\mathscr{U} \subseteq \{\bar{n}\sigma/n \in \omega\}$, while a unary mapping Σ is *universal for* $\mathscr{M}$ iff $\mathscr{M} \subseteq \{\lambda\theta.\bar{n}\Sigma(\theta)/n \in \omega\}$. We consider unary mappings since the relevant definitions and results for n-ary mappings are obtained in an obvious way from 7.20. Notice that $\mathscr{U}$ (respectively $\mathscr{M}$) is countable whenever it admits a universal element (mapping).

Proposition 9.17. Enumeration Lemma. Let $\mathscr{B}$ be finite. Then there is an element τ recursive in $\mathscr{B}$ and universal for the elements polynomial in $\mathscr{B}$.

Proof. Let $\{L, R\} \cup \mathscr{B} = \{\psi_0, \ldots, \psi_m\}$ and $\mathscr{U} = cl(\psi_0, \ldots, \psi_m/\circ, \Pi)$. It follows by 4.13 that

$$\mathscr{U} = cl(\psi_0, \ldots, \psi_m/\lambda\theta.\psi_i\theta, i \leq m, \Pi).$$

Take $\psi = (\psi_0, \ldots, \psi_m, L)$, $\chi = (\langle\psi_0\rangle, \ldots, \langle\psi_m\rangle, L)$ and $\tau = R[Q(\psi L, Q\chi R, QTR, L)]$. To show that τ is universal for $\mathscr{U}$, we shall use the equality $\tau = Q(\psi, Q\chi\tau, QT\tau, I)$ and the fact that $\overline{J(s,t)}Q = \bar{s}\bar{t}$ by exercise 6.6, where $J = \lambda st.(2s+1)2^t$.

If $i \leq m$, then

$$\overline{J(i,0)}\tau = \bar{i}\bar{0}(\psi, Q\chi\tau, QT\tau, I) = \bar{i}\psi = \psi_i.$$

Let $i \leq m$ and $\varphi = \bar{k}\tau$. Then

$$\overline{J(J(k,i),1)}\tau = \overline{J(k,i)}Q\chi\tau = \bar{k}\langle\psi_i\rangle\tau = \psi_i\bar{k}\tau = \psi_i\varphi.$$

Let $\varphi = \bar{k}\tau$, $\psi = \bar{l}\tau$. Then

$$\overline{J(J(k,l),2)}\tau = \bar{k}\bar{l}T\tau = (\bar{k}\tau, \bar{l}\tau) = (\varphi, \psi)$$

by 7.14. We are done.

The following Enumeration Theorem is an analog to theorem XXII of Kleene [1952] and the corresponding statement of Skordev [1980].

Proposition 9.18. Let $\mathscr{B}$ be finite and $\mathscr{U}$ be the set of all elements recursive in $\mathscr{B}$. Then there is an element $\sigma \in \mathscr{U}$ which is universal for $\mathscr{U}$.

Proof. There exists by 9.17 an element $\tau \in \mathscr{U}$ universal for the elements polynomial in $\mathscr{B}_0 \cup \langle\mathscr{B}\rangle$. Take $\rho = C([I]L, R)$ and $\sigma = \langle\bar{1}\rangle\rho[D\langle\tau\rangle\rho]$. It follows that $\sigma \in \mathscr{U}$ and $\bar{n}\sigma = \bar{1}[\bar{n}\tau]$ for all n by 6.43.

Let $\varphi \in \mathscr{U}$. Then 9.3 implies that $\varphi = \bar{1}[\psi]$ with a certain ψ strictly primitive in $\mathscr{B}$, hence polynomial in $\mathscr{B}_0 \cup \langle\mathscr{B}\rangle$. There is a number n such that $\psi = \bar{n}\tau$; hence

$$\varphi = \bar{1}[\psi] = \bar{1}[\bar{n}\tau] = \bar{n}\sigma.$$

Therefore, σ is universal for $\mathscr{U}$. We are done.

Alternatively, one may use in the above proof the equivalence (1)$\Leftrightarrow$(2) of 7.11 and 9.2 instead of 9.3.

The following statement is called the Parametrized Enumeration Theorem.

Proposition 9.19. Let $\mathscr{B}$ be finite and $\mathscr{M}$ be the set of all unary mappings recursive in $\mathscr{B}$. Then there is a mapping $\Sigma \in \mathscr{M}$ which is universal for $\mathscr{M}$.

The proof follows that of 9.18. Just take

$$\Sigma = \lambda\theta.\langle \bar{1} \rangle \rho[D\langle \tau(I, \langle \theta \rangle)\rangle \rho]$$

and make use of the Normal Form Theorem 9.4.

Assume from now on that a subset $\mathscr{B}$ of $\mathscr{F}$ is fixed, $\mathscr{U}$ is the set of all elements recursive in $\mathscr{B}$ and $\mathscr{M}$ is the set of all unary mappings recursive in $\mathscr{B}$. As in the proof of 9.17, J will stand for the pairing function $\lambda st.(2s+1)2^t$.

An element ψ is *natural* iff for all s there is a t such that $\bar{s}\psi = \bar{t}$. In other words, natural elements are those which represent total unary number theoretic functions.

An element σ is *principal universal for* $\mathscr{U}$ iff for all $\varphi \in \mathscr{U}$ there is a natural primitive recursive element ψ such that $\bar{n}\varphi = \bar{n}\psi\sigma$ for all n. A mapping Σ is *principal universal for* $\mathscr{M}$ iff for all $\Gamma \in \mathscr{M}$ there is a natual primitive recursive element ψ such that $\bar{n}\Gamma(\theta) = \bar{n}\psi\Sigma(\theta)$ for all n, θ. We shall concentrate on principal universal elements since the corresponding statements for principal universal mappings may be obtained by an immediate parametrization.

Proposition 9.20. If σ is principal universal for $\mathscr{U}$, then σ is universal for $\mathscr{U}$.

Proof. Let $\varphi \in \mathscr{U}$. Then $[I]\varphi \in \mathscr{U}$; hence there is a natural element ψ such that $\bar{n}[I]\varphi = \bar{n}\psi\sigma$ for all n. It follows that $\bar{0}\psi = \bar{k}$ for a certain k, hence

$$\varphi = \bar{0}[I]\varphi = \bar{0}\psi\sigma = \bar{k}\sigma.$$

This completes the proof.

Proposition 9.21. If σ is universal for $\mathscr{U}$, then $Q\sigma$ is principal universal for $\mathscr{U}$.

Proof. Let $\varphi \in \mathscr{U}$. Then $\varphi = \bar{k}\sigma$ with a certain k. Taking $\psi = \bar{k}P$, we get $\bar{n}\psi = \overline{J(n,k)}$ for all n; hence ψ is a natural primitive recursive element. It follows that

$$\bar{n}\varphi = \bar{n}\bar{k}\sigma = \overline{J(n,k)}Q\sigma = \bar{n}\bar{k}PQ\sigma = \bar{n}\psi Q\sigma$$

for all n, hence $Q\sigma$ is principal universal for $\mathscr{U}$. The proof is complete.

The following statement is called the Second Recursion Theorem. It is an analogue to theorem XXVII of Kleene [1952] and the corresponding theorem of Skordev [1980].

Proposition 9.22. If $\sigma \in \mathscr{U}$ is principal universal for $\mathscr{U}$ and $\varphi \in \mathscr{U}$, then there is an n such that $\bar{n}\varphi = \bar{n}\sigma$.

Proof. We have $Q\sigma \in \mathscr{U}$; hence there is a natural primitive recursive element ψ such that $\bar{n}Q\sigma = \bar{n}\psi\sigma$ for all n. Proposition 9.20 implies that $DP\psi\varphi = \bar{m}\sigma$ for a certain m since $DP\psi\varphi \in \mathscr{U}$. Taking $\bar{n} = \overline{J(m,m)}\psi$, we get

$$\bar{n}\varphi = \bar{m}\bar{m}P\psi\varphi = \bar{m}DP\psi\varphi = \bar{m}\bar{m}\sigma = \overline{J(m,m)}Q\sigma = \overline{J(m,m)}\psi\sigma = \bar{n}\sigma,$$

which completes the proof.

A parametrized version of the Second Recursion Theorem states that whenever $\Sigma \in \mathscr{M}$ is principal universal for $\mathscr{M}$ and $\Gamma \in \mathscr{M}$ then there is an n such that $\bar{n}\Gamma(\theta) = \bar{n}\Sigma(\theta)$ for all θ.

The Enumeration Theorem and the Second Recursion Theorem imply a fixed point result called the Recursion Theorem.

Proposition 9.23. If Γ is a uniary mapping recursive in $\mathscr{B}$, then it has a fixed point φ recursive in $\mathscr{B}$.

Proof. The mapping Γ is recursive in a finite subset $\mathscr{B}_1$ of $\mathscr{B}$. Let Γ^* correspond to Γ by 7.21 and let σ be recursive in $\mathscr{B}_1$ and principal universal for the elements recursive in $\mathscr{B}_1$; such an element σ exists by 9.18, 9.21.

The element $\Gamma^*(\sigma)$ is recursive in $\mathscr{B}_1$; hence there is an n such that $\bar{n}\Gamma^*(\sigma) = \bar{n}\sigma$ by 9.22. Taking $\varphi = \bar{n}\sigma$, we get

$$\Gamma(\varphi) = \Gamma(\bar{n}\sigma) = \bar{n}\Gamma^*(\sigma) = \bar{n}\sigma = \varphi,$$

which completes the proof.

The element φ so constructed is not necessarily a least fixed point of Γ; hence 9.23 is not a First Recursion Theorem. However, a recursive fixed point of a given mapping is often all one needs. The advantage of the above Recursion Theorem is that it depends on no μ-axiom stronger than μA_0.

A parametrized version of 9.23 states that whenever Γ is a $n+1$-ary mapping recursive in $\mathscr{B}$, then there is a n-ary mapping Γ_1 recursive in $\mathscr{B}$ such that

$$\Gamma(\theta_1, \ldots, \theta_n, \Gamma_1(\theta_1, \ldots, \theta_n)) = \Gamma_1(\theta_1, \ldots, \theta_n)$$

for all $\theta_1, \ldots, \theta_n$. Systems of equalities may also be solved.

The following statement is an analogue to a theorem of Rice [1953].

Proposition 9.24. Let $\sigma \in \mathscr{U}$ be principal universal for $\mathscr{U}$ and $\emptyset \subset \mathscr{E} \subset \mathscr{U}$. Then there is no natural element $\psi \in \mathscr{U}$ such that $\bar{n}\sigma \in \mathscr{E}$ iff $\bar{n}\psi = \bar{0}$.

Proof. Suppose there is such an element $\psi \in \mathscr{U}$. Take k, l such that $\bar{k}\sigma \in \mathscr{E}$, $\bar{l}\sigma \in \mathscr{U} \setminus \mathscr{E}$, then take $\tau = \psi(\bar{l}, [I]\bar{k})$. It follows that $\tau\sigma \in \mathscr{U}$, hence $\bar{n}\tau\sigma = \bar{n}\sigma$ for a certain n by the Second Recursion Theorem.

If $\bar{n}\psi = \bar{0}$, then $\bar{n}\sigma \in \mathscr{E}$, $\bar{n}\tau = \bar{l}$, hence

$$\bar{l}\sigma = \bar{n}\tau\sigma = \bar{n}\sigma \in \mathscr{E},$$

which is not the case.

If $\bar{n}\psi \neq \bar{0}$, then $\bar{n}\sigma \notin \mathscr{E}$, $\bar{n}\tau = \bar{k}$, hence $\bar{k}\sigma = \bar{n}\tau\sigma = \bar{n}\sigma \notin \mathscr{E}$, which again is not the case. The proof is complete.

We close our study of universal elements and mappings by a discussion of the Theory of Numberings. Our reasons for doing this are first to avoid any overwork and, secondly, the proofs in the general theory turn out to be adaptations of the original proofs of Ordinary Recursion Theory, see 9.22, 9.24 for example.

To each element $\sigma \in \mathscr{U}$ universal for $\mathscr{U}$ assign a *numbering* σ^* of $\mathscr{U}$ by the equality

$$\sigma^* n = \bar{n}\sigma, n \in \omega.$$

Take the set Ψ_0 of all unary partial number theoretic functions weakly representable by recursive elements. Proposition 8.2 implies that all unary partial recursive functions are in Ψ_0 and if f is a unary function μ-recursive in Ψ_0, then $f \in \Psi_0$. Obviously, there are nonrecursive functions in Ψ_0.

Proposition 9.25. Let $\sigma \in \mathscr{U}$ be principal universal for $\mathscr{U}$. Then σ^* is a *precomplete numbering* of $\mathscr{U}$ in the sense of Maltsev [1961], where the last notion is relativized by considering members of Ψ_0 instead of unary partial recursive functions.

Proof. Let $f \in \Psi_0$. Then f is weakly represented by a recursive element φ. It follows that $\varphi\sigma \in \mathscr{U}$; hence there is a natural primitive recursive element ψ such that $\bar{n}\varphi\sigma = \bar{n}\psi\sigma$ for all n. The element ψ represents a total function $g \in \Psi_0$ and if $f(n)\downarrow$, then

$$\sigma^* g(n) = \overline{g(n)}\sigma = \bar{n}\psi\sigma = \bar{n}\varphi\sigma = \overline{f(n)}\sigma = \sigma^* f(n).$$

It now follows from a statement of Ershov [1977] that σ^* is a precomplete numbering. This completes the proof.

It is worth mentioning that the arguments in Ershov's book concerning precomplete numberings depend on the closure properties of the class of all unary partial recursive functions, so they remain valid for the above relativized notion of precomplete numbering.

Proposition 9.26*.** If $\sigma \in \mathscr{U}$ is principal universal for $\mathscr{U}$, then σ^* is a *complete numbering* in the sense of Maltsev [1965].

Proof. Let f be a unary partial number theoretic function. Then f is represented by a recursive element φ by 8.3***. It follows again that $\varphi\sigma \in \mathscr{U}$; hence there is a natural primitive recursive element ψ such that $\bar{n}\varphi\sigma = \bar{n}\psi\sigma$ for all n. The element ψ represents a general recursive function g by exercise 8.5***.

If $f(n)\downarrow$, then $\sigma^* g(n) = \sigma^* f(n)$ as in the proof of 9.25.

We have $O \in \mathscr{U}$ and whenever $f(n)\uparrow$, then $\bar{n}\varphi = O$; hence

$$\sigma^* g(n) = \overline{g(n)}\sigma = \bar{n}\psi\sigma = \bar{n}\varphi\sigma = O\sigma = O.$$

Therefore, σ^* is a complete numbering. The proof is complete.

As a corollary to 9.25 we get the following analogue to a theorem of Rogers [1967].

Proposition 9.27. Let $\sigma, \sigma_1 \in \mathscr{U}$ be principal universal for $\mathscr{U}$. Then there is a natural recursive element φ such that $\bar{n}\sigma = \bar{n}\varphi\sigma_1$ for all n and φ represents a permutation of ω, i.e. $\bar{m}\varphi \neq \bar{n}\varphi$ whenever $m \neq n$, and for all n there is an m such that $\bar{m}\varphi = \bar{n}$.

Proof. Since $\sigma \in \mathscr{U}$, there is a natural primitive recursive element ψ such that $\bar{n}\sigma = \bar{n}\psi\sigma_1$ for all n. The element ψ represents a total function $g \in \Psi_0$ and it follows that

$$\sigma^* n = \bar{n}\sigma = \bar{n}\psi\sigma_1 = \overline{g(n)}\sigma_1 = \sigma_1^* g(n)$$

for all n, hence g reduces σ^* to σ_1^*. Similarly, the numbering σ_1^* is reducible to σ^* by a total function from Ψ_0. Using a (relativized) theorem of Maltsev [1961], we conclude that σ^* and σ_1^* are recursively in Ψ_0 isomorphic, i.e., there is a μ-recursive in Ψ_0 function f such that $\sigma^* n = \sigma_1^* f(n)$ for all n and f is a permutation of ω. Then $f \in \Psi_0$; hence there is a recursive element φ to weakly represent f. However, f is total, hence φ represents f and

$$\bar{n}\sigma = \sigma^* n = \sigma_1^* f(n) = \overline{f(n)}\sigma_1 = \bar{n}\varphi\sigma_1$$

for all n. This completes the proof.

Proposition 9.27 may also be proved directly by making use of the Recursion Theorem 9.23. A parametrized version can be established, too.

EXERCISES TO CHAPTER 9

Exercise 9.1. Give an alternative proof of 9.10 based on 6.16 and exercise 6.2 instead of exercise 6.1i and proposition 6.30.

Hint. Show that $\mu\theta.\varphi[\psi(I,\theta\chi)] = \varphi\bar{1}[\sigma]$, where $\sigma = \mu\theta.((\bar{0},\psi(\bar{2},\varphi\bar{4})),\bar{1}, \chi\bar{2},\theta R^3)$. To that end, supposing $\varphi[\psi(I,\tau\chi)] \leq \tau$ and writing ρ for $[\psi(I,\tau\chi)]$, prove that $R[\sigma] \leq \mu\theta.(\rho,\rho,\chi\rho,\theta\chi\rho)$. (If μA_1 is allowed, then one does not need exercise 6.2 to get the last inequality.)

Exercise 9.2. Let $I^* \in \mathscr{F}$ satisfy condition (£)* of exercise 6.8*. Using the Parametrized Recursion Theorem and exercises 6.1h, 6.10, give an alternative proof of 9.12*.

Hint. The operation Δ^* of exercise 6.10 exists and is recursive by the Parametrized Recursion Theorem and exercise 7.1. Take

$$\sigma = \Delta^*(((\bar{0},\psi(R\bar{1},\langle\varphi\bar{0}\bar{1}\rangle C(R\bar{1},R^2))),\bar{0}\bar{1}),C(R\bar{1},R^2)).$$

Using exercise 6.1h, show that $\varphi\bar{0}\bar{1}[\sigma]$ is a solution of $\varphi[\psi(I,\langle\theta\rangle)] = \theta$. Supposing $\varphi[\psi(I,\langle\tau\rangle)] \leq \tau$, take $\rho = [\psi(I,\langle\tau\rangle)]$ and prove that $R[\sigma] \leq \Delta^*((\rho,\rho),\rho)$.

Combining 9.4 and exercise 9.2 one gets a second proof of the First Recursion Theorem which depends on μA_0 plus (£)*. Another proof given in Ivanov [1980] assumes that $\langle\ \ \rangle$, $[\ \ ]$ satisfy the requirements of 5.13; there is also a proof due to J. Zashev which depends on similar assumptions and uses an ordinary coding of terms.

The following exercise establishes a parametrized version of 9.16*.

Exercise 9.3*. Let a system of inequalities

$$\Gamma_i(\theta_1,\ldots,\theta_{n+m}) \leq \theta_{n+i}, \quad 1 \leq i \leq m,$$

be given, and let $\Gamma_1,\ldots,\Gamma_m$ be recursive in $\mathscr{B}$. Show that there are n-ary mappings $\Gamma_1^*,\ldots,\Gamma_m^*$ recursive in $\mathscr{B}$ such that

$$\Gamma_i(\theta_1,\ldots,\theta_n,\Gamma_1^*(\theta_1,\ldots,\theta_n),\ldots,\Gamma_m^*(\theta_1,\ldots,\theta_n)) = \Gamma_i^*(\theta_1,\ldots,\theta_n)$$

for $1 \leq i \leq m$ and all $\theta_1,\ldots,\theta_n$, and if $\tau_1,\ldots,\tau_{n+m}$ satisfy the above system, then $\Gamma_i^*(\tau_1,\ldots,\tau_n) \leq \tau_{n+i}$, $1 \leq i \leq m$.

If the element U of exercise 7.8 is available, then even more general systems of inequalities can be solved. The idea of using least upper bounds is due to J. Zashev.

Exercise 9.4. Let U satisfy condition (1) of exercise 7.8 and suppose given a system of inequalities

$$\Gamma_j(\theta_0,\dots,\theta_n) \leq \theta_{i_j}, \quad j \leq m,$$

with $i_0,\dots,i_m \leq n$ and suppose that $\Gamma_0,\dots,\Gamma_m$ are recursive in $\{U\}\cup\mathscr{B}$. Transform this system into an equivalent one introduced by mappings recursive in $\{U\}\cup\mathscr{B}$ and such that each θ_i appears at most once on the right.

Hint. Show that each pair of inequalities

$$\Gamma_k(\theta_0,\dots,\theta_n) \leq \theta_i, \quad \Gamma_l(\theta_0,\dots,\theta_n) \leq \theta_i$$

is equivalent to the single inequality

$$U(\Gamma_k(\theta_0,\dots,\theta_n), \quad \Gamma_l(\theta_0,\dots,\theta_n)) \leq \theta_i.$$

It is worth mentionning that the least solution turns an inequality $\Gamma_j(\theta_0,\dots,\theta_n) \leq \theta_{i_j}$ into an equality, provided θ_{i_j} occurs just once on the right side of the given system.

Let $\mathscr{B} \subseteq \mathscr{F}$ and $\mathscr{U}$ (respectively $\mathscr{M}$, stand for the set of all the elements (unary mappings) recursive in $\mathscr{B}$.

Exercise 9.5. Following the proof of 9.17, prove the Enumeration Theorem by means of the Recursion Theorem.

Hint. Let $\mathscr{B} = \{\psi_1,\dots,\psi_m\}$, $\psi = (L, R, \psi_1,\dots,\psi_m, L)$ and $\rho = C([I]L, R)$. The mapping

$$\Gamma = \lambda\theta.Q(\psi, Q\langle\theta\rangle\theta, QT\theta, G\langle\theta\rangle, \rho[D\langle\theta\rangle\rho], L)$$

is recursive in $\mathscr{B}$, hence it has a fixed point σ recursive in $\mathscr{B}$. Show that σ is universal for $\mathscr{U}$.

Exercise 9.6. Show that whenever Σ is universal for $\mathscr{M}$, then $\Sigma(\psi)$ is universal for the elements recursive in $\{\psi\}\cup\mathscr{B}$. In particular, $\Sigma(I)$ is universal for $\mathscr{U}$.

Exercise 9.7. Show that $\mathscr{U}$ (respectively $\mathscr{M}$) has a universal element $\sigma\in\mathscr{U}$ (mapping $\Sigma\in\mathscr{M}$) iff the set $\mathscr{B}$ is finitely generated, i.e. all the members of $\mathscr{B}$ are recursive in a finite subset $\mathscr{B}_1$ of $\mathscr{B}$.

Both the Recursion Theorem 9.23 and its parametrized version can be reduced by 7.20 to the following assertion: Whenever Γ_1 is a unary mapping recursive in $\mathscr{B}$, then there is a unary mapping Γ recursive in $\mathscr{B}$ such that $\Gamma_1((\theta, \Gamma(\theta))) = \Gamma(\theta)$ for all θ. One may put the question whether more complicated equalities can be solved. For instance, is there a mapping Γ recursive in $\mathscr{B}$ such that $\Gamma_1((\theta, \Gamma(\Gamma_2(\theta)))) = \Gamma(\theta)$ for all θ, provided Γ_1, Γ_2 are recursive in $\mathscr{B}$?

Speaking somewhat informally, let $\Gamma_1,\dots,\Gamma_m$ be unary mappings over $\mathscr{F}$, let Γ stand for a unary mapping and let $\mathscr{V}(\theta, \Gamma)$ be an expression constructed by composing $\Pi, \Gamma, \Gamma_1, \ldots, \Gamma_m$. (The pairing operation is added to compensate for the unarity of the mappings.) The following exercise establishes a Generalized Recursion Theorem.

Exercise 9.8. Let $\Gamma_1, \ldots, \Gamma_m$ be recursive in $\mathscr{B}$. Prove that there is a mapping Γ recursive in $\mathscr{B}$ such that $\mathscr{V}(\theta, \Gamma) = \Gamma(\theta)$ for all θ.

Hint. The mappings $\Gamma_1, \ldots, \Gamma_m$ are recursive in a finite subset $\mathscr{B}_1$ of $\mathscr{B}$. Take a mapping Σ recursive in $\mathscr{B}_1$ and principal universal for the unary mappings recursive in $\mathscr{B}_1$, and consider the expression $\mathscr{V}_1(\theta, \theta_1, \Gamma)$ obtained from $\mathscr{V}(\theta, \Gamma)$ by substituting $\theta_1 \Gamma$ for Γ. The mapping $\Gamma^* = \lambda\theta\theta_1 . \mathscr{V}_1(\theta, \theta_1, \Sigma)$ is recursive in $\mathscr{B}_1$; hence there is by the Parametrized Transition Theorem a mapping Γ^{**} recursive in $\mathscr{B}_1$ such that $\bar{n}\Gamma^{**}(\theta, \theta_1) = \Gamma^*(\theta, \bar{n}\theta_1)$ for all n, θ, θ_1. There is n such that $\bar{n}\Gamma^{**}(\theta, I) = \bar{n}\Sigma(\theta)$ for all θ by the Parametrized Second Recursion Theorem. Taking $\Gamma = \lambda\theta . \bar{n}\Sigma(\theta)$, one gets

$$\mathscr{V}(\theta, \Gamma) = \mathscr{V}_1(\theta, \bar{n}, \Sigma) = \Gamma^*(\theta, \bar{n}) = \bar{n}\Gamma^{**}(\theta, I) = \bar{n}\Sigma(\theta) = \Gamma(\theta)$$

for all θ, which completes the proof.

The above theorem can obviously be restated for n-ary mappings as follows. If $\Gamma_1, \ldots, \Gamma_m$ are mappings over $\mathscr{F}$ recursive in $\mathscr{B}$, Γ stands for a n-ary mapping and the expression $\mathscr{V}(\theta_1, \ldots, \theta_n, \Gamma)$ is constructed by composing $\Gamma, \Gamma_1, \ldots, \Gamma_m$, then there is a mapping Γ recursive in $\mathscr{B}$ such that $\mathscr{V}(\theta_1, \ldots, \theta_n, \Gamma) = \Gamma(\theta_1, \ldots, \theta_n)$ for all $\theta_1, \ldots, \theta_n$.

A generalized First Recursion Theorem established in the exercises to chapter 12 will show that moreover, the equalities considered above have solutions Γ recursive in $\mathscr{B}$ which are least with respect to the pointwise partial order.

Exercise 9.9. Let $\Gamma: \mathscr{F} \to \mathscr{F}$ be recursive in $\mathscr{B}$. Construct a mapping Γ^* recursive in $\mathscr{B}$ such that $\bar{n}\Gamma^*(\theta) = \Gamma^n(\theta)$ for all n, θ.

Hint. Use exercise 9.8 to obtain a mapping Γ^* recursive in $\mathscr{B}$ such that $\Gamma^*(\theta) = (\theta, \Gamma^*(\Gamma(\theta)))$ for all θ.

The following exercise strengthens both the Second Recursion Theorem and the Recursion Theorem.

Exercise 9.10. Let $\sigma \in \mathscr{U}$ be principal universal for $\mathscr{U}$. Prove that for every binary mapping Γ recursive in $\mathscr{B}$ there is a natural primitive recursive element ψ such that $\Gamma(\bar{n}\psi, \bar{n}) = \bar{n}\psi\sigma$ for all n.

Hint. Applying the Parametrized Transition Theorem twice, obtain a $\rho \in \mathscr{U}$ such that $\Gamma(\bar{m}, \bar{n}) = \bar{m}\bar{n}\rho$ for all m, n. Take natural primitive recursive elements ψ_0, ψ_1 such that $\bar{n}Q\sigma = \bar{n}\psi_0\sigma, \bar{n}\langle DP\psi_0 \rangle \rho = \bar{n}\psi_1\sigma$ for all n, then take $\psi = \psi_1 DP\psi_0$ and show that $\bar{n}\psi\bar{n}\rho = \bar{n}\psi\sigma$ for all n.