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L. L. Ivanov

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ALGEBRAIC RECURSION THEORY

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ELLIS HORWOOD LIMITED
Publishers · Chichester

Halsted Press: a division of
JOHN WILEY & SONS
New York · Chichester · Brisbane · Toronto

First published in 1986 by
ELLIS HORWOOD LIMITED
Market Cross House, Cooper Street,
Chichester, West Sussex, PO19 1EB, England
The publisher's colophon is reproduced from James Gillison's drawing of the ancient Market Cross, Chichester.

Distributors:
Australia and New Zealand:
JACARANDA WILEY LIMITED
GPO Box 859, Brisbane, Queensland 4001, Australia
Canada:
JOHN WILEY & SONS CANADA LIMITED
22 Worcester Road, Rexdale, Ontario, Canada
Europe and Africa:
JOHN WILEY & SONS LIMITED
Baffins Lane, Chichester, West Sussex, England
North and South America and the rest of the world:
Halsted Press: a division of
JOHN WILEY & SONS
605 Third Avenue, New York, NY 10158, USA

British Library Cataloguing in Publication Data
Ivanov, L.L.
Algebraic recursion theory. —
(Mathematics and its applications)
1. Recursive functions
I. Title II. Bell, J.L. III. Series
511.3 QA9.615

Library of Congress Card No. 86–20923

ISBN 0–7458–0102–1 (Ellis Horwood Limited)
ISBN 0–470–20725–6 (Halsted Press)

Printed in Great Britain by The Camelot Press, Southampton

Preface

This monograph on algebraic recursion theory has its origins in a disertation submitted to Sofia University in 1980. Numerous recent developments are included and, as is often the case, the very process of working on the manuscript added new episodes to the story.

Undoubtedly, the fundamental works of D. Skordev on algebraic recursion theory have made this book possible. Skordev's mathematical ideas and style of thinking have had a significant effect on the whole of my work and I am deeply indebted to him for his invaluable help and encouragement.

The papers of S. C. Kleene on higher recursion theory are of particular importance for me since the present approach was inspired by the idea of embracing Kleene's notion of recursiveness in higher types. The works of Y. N. Moschovakis on inductive definability and recursion in higher types should also be mentioned on acccount of their beneficial influence.

Acknowledgements:
I should like to acknowledge my colleagues D. Vakarelov, N. Georgieva, J. Zashev. I. Soskov, S. Passay, T. Tinchev, S. Radev, A. Buda, P. Petkov, G. Gargov and V. Sotirov for their helpful comments and discussions.

I thank Dr. J. L. Bell of London School of Economics who polished up the 'Bulgarian English' of this book; and also Mr Ellis Horwood, Mrs Patricia Morrison and Mrs Hazel Kelly, who made my work with Ellis Horwood Publishers a real pleasure.

The final version of the book was prepared during my stay at Oslo University during the spring term 1985. I am indebted to my Norwegian colleagues and especially to D. Normann for stimulating discussions.

Sofia, September 1986 L. L. Ivanov

To Pepa and Borislava

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PART A

Introduction

CHAPTER 1

Preliminaries

This book aims to provide a straightforward introduction to some fundamentals of modern Recursion Theory and to establish a basis for further work in this and related areas. The mathematical core of the subject is the theory of an algebraic system which we call an *iterative operative space*, IOS for short, which is a partially ordered set with four basic operations satisfying several axioms.

The algebraic system in question has been specially designed to support a unified yet selective axiomatic study of the concept of effective computability arising in a wide range of disciplines, from Computer Science to Generalized Recursion Theory. Some reasons for our choice are given in the introductory chapter 2, while a further discussion follows in the Epilogue.

By way of illustration two standard spaces are presented in chapter 3. They are essentially different, so the axiomatic system of IOS has no single standard model to reflect adequately its intuitive aspects. Nonetheless, one could start with some particular spaces and only later pass to the general theory. It seems however that the detailed preliminary or parallel study of examples is not a very good introduction, for the key to the theory lies in the μ-induction technique which is equally clear in particular spaces and *in abstract*. Perhaps the best way to familiarize oneself with IOS is to practise proving properties of its initial operations as done in chapters 4, 6. Finally, passing from the general to the particular is in closer conformity with our traditional Bulgarian way of thinking; for instance, we address a letter to South Georgia, Grytviken, Scotia Arc College, not Scotia Arc College, Grytviken, South Georgia, as is the custom in other countries. These are the chief reasons for arranging the exposition as we have.

The book consists of seven parts, part A being the Introduction. In part B a general theory of recursion is developed. Abstract notions of primitive recursiveness and recursiveness are introduced, a number of properties of the IOS-operations are established and used to obtain some basic statements including a Pull Back Theorem, a Transition Theorem, Representation Theorems for primitive recursive and partial recursive number theoretic functions, Normal Form Theorems, Enumeration Theorems, a Kleene First Recursion Theorem, a Second Recursion Theorem, abstract Rice and Rogers Theorems.

Part C broadens the argument with recursion theory on so called consecutive spaces, spaces with additional operations and hierarchies of spaces. The hierarchies in question are transfinite sequences of consecutive IOS each of which is a proper subspace of its successors. In particular, abstract analogues to Platek's First Recursion Theorem are established. Thus parts B, C introduce the reader straightaway to the central topics of the book. To keep fiction in touch with reality certain examples of IOS are regularly used as sources of illustration, mainly in the exercises.

Part D deals with some constructions yielding IOS, consecutive IOS and hierarchies of IOS.

In part E a series of spaces is used to express particular notions of effective computability, e.g. the relative μ-recursiveness and partial recursiveness of Ordinary Recursion Theory, the stack recursiveness of Germano and Maggiolo–Schettini [1976], the prime and search computability of Moschovakis [1969], as well as computabilities by abstract programs. It is not our intention to present or study all known spaces and notions of effective computability but rather to suggest typical patterns of modelling applicable to other similar situations as well.

Connections with certain related theories are established in part F. The system of de Bakker and Scott, better known as Scott's μ-calculus, and those of Skordev [1982, 1976] are considered in chapters 26, 27. In chapters 28–30 appropriate spaces provide a framework for the theory of recursive functionals of Kechris and Moschovakis [1977], Higher Recursion Theory and Inductive Definability Theory. While parts E, F clarify the way in which the general axiomatic theory produces particular recursion theories, few details about these are given, for this is beyond the scope of the book. Works listed in the References offer further introduction and bibliography; especially useful is Barwise [1977], part C.

Part G comprises some final remarks on the axiomatization of Recursion Theory with special attention paid to the present approach.

The exposition is necessarily detailed since no comprehensive presentation of the subject can be found elsewhere. Nonetheless, the author has tried to clear a middle course between an encyclopaedic reference book and a readable text-book. A number of exercises have been provided, sometimes contributing to the main text.

A lecture course based on the book may include a minimum of general theory and Ordinary Recursion Theory (selected parts of chapters 4–9, 18, 19, 22) as applied to a more specific area, say the Mathematical Theory of Programs (chapters 23, 26) or Computable Functions over Arbitrary Domains (chapters 10, 21, 24), Recursion in Higher Types (chapters 28, 29). Inductive Definability (chapter 30), a corresponding introduction to Recursion on Set Functions or Ordinal Functions etc. If the field of further interest is to be Algebraic Recursion Theory itself, then substantial parts of chapters 10, 12–16, 20, 27 should be included instead.

Because of its axiomatic character the present work is essentially self-contained. The reader is assumed to possess the ordinary mathematical sophistication of an undergraduate student and a slender acquaintance with ordinals and transfinite induction.

Some standard notations are used throughout the book. 'Iff' is, as usual, written for 'if, and only if'. The letter ω denotes the set of all natural numbers (nonnegative integers), as well as the first infinite cardinal. Letters m, n, i, j, k, l stand for natural numbers and ξ, η, ζ for ordinals.

If M, N are sets, then $f: M \dashrightarrow N$ means that f is a partial *single-valued function* from M to N, i.e. $f \subseteq M \times N$ and whenever (s,t), $(s,r) \in f$, then $r = t$. Symbols $\downarrow$ and $\uparrow$ are written respectively for 'is defined' and 'is not defined'. The equality sign is also used as a sign of conventional equality. For instance, if $f, g: M \dashrightarrow N$, then $f(s) = g(s)$ means that either $f(s)$, $g(s)$ are defined and identical, or $f(s)\uparrow, g(s)\uparrow$ (the usual notation is $f(s) \simeq g(s)$). If f is a partial *multiple-valued function* from M to N, $f: M \dashrightarrow 2^N \setminus \{\varnothing\}$, then we write also $f: M \to 2^N$ or $f \subseteq M \times N$, meaning $f(s)\uparrow$ if $f(s) = \varnothing$ and identifying f with its graph. A countable sequence $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$ is written $\{\varphi_n\}_{n \in \omega}$ or just $\{\varphi_n\}$, and similarly $\{\varphi_\xi\}$ is a transfinite sequence.

If $\mathscr{F}$ is a set, $\mathscr{B} \subseteq \mathscr{F}$ and $\mathscr{B}'$ is a set of operations over $\mathscr{F}$ (i.e. mappings $\Gamma: \mathscr{F}^n \to \mathscr{F}, n > 0$), then $cl(\mathscr{B}/\mathscr{B}')$ stands for the *closure* of $\mathscr{B}$ under the operations in $\mathscr{B}'$. Formally, this is the least set $\mathscr{B}^*$ satisfying the following two conditions:

1. $\mathscr{B} \subseteq \mathscr{B}^*$.
2. If $\Gamma: \mathscr{F}^n \to \mathscr{F} \in \mathscr{B}'$, then $\Gamma(\mathscr{B}^{*n}) \subseteq \mathscr{B}^*$.

In other words, $cl(\mathscr{B}/\mathscr{B}')$ consists of those members of $\mathscr{F}$ which can be obtained from members of $\mathscr{B}$ by means of operations from $\mathscr{B}'$. Notice that whenever $\varphi \in cl(\mathscr{B}/\mathscr{B}')$, then there is a finite subset $\mathscr{B}_0$ of $\mathscr{B}$ such that $\varphi \in cl(\mathscr{B}_0/\mathscr{B}')$.

If $\mathscr{F}$ is a partially ordered set and $\Gamma: \mathscr{F} \to \mathscr{F}$, we write $\mu\theta.\Gamma(\theta)$ for the least solution of the inequality $\Gamma(\theta) \leq \theta$, if it exists. If Γ is monotonic, then $\theta_0 = \mu\theta.\Gamma(\theta)$ is also the *least fixed point* of Γ. Though the argument is immediate and well known, here it is. Indeed, it suffices to show that θ_0 is a fixed point of Γ. The inequality $\Gamma(\theta_0) \leq \theta_0$ implies $\Gamma(\Gamma(\theta_0)) \leq \Gamma(\theta_0)$ by the monotonicity of Γ; hence $\Gamma(\theta_0)$ is another solution to the inequality $\Gamma(\theta) \leq \theta$. This gives $\theta_0 \leq \Gamma(\theta_0)$, hence $\Gamma(\theta_0) = \theta_0$.

All the statements in the book are called propositions and enumerated uniformly. For instance, 5.11 refers to the eleventh statement of chapter 5. The exercises and examples are enumerated separately.

CHAPTER 2

Effective computability

What is *effective computability*?

Broadly speaking, a class of elements $\mathscr{F}$ is given, where the elements may be functions, functionals, operators, or any other kind of 'data processing' objects. Some elements are taken as initial because of their 'obvious effective computability'. Certain initial 'effective' operations over $\mathscr{F}$ are also chosen, assuming they preserve 'effective computability'. New 'effectively computable' elements are generated from initial ones by means of initial operations and their properties are studied.

A classical example of a class of effectively computable objects is the effectively computable number theoretic functions studied in Ordinary Recursion Theory. According to Church's thesis (which is widely accepted) these are exactly the *partial recursive functions*, which can be defined inductively as follows.

1. The functions $\lambda s.0$, $\lambda s.s+1$ and $\lambda s_1 \dots s_n.s_i$, $1 \leq i \leq n$, are partial recursive.
2. If $f:\omega^m \dashrightarrow \omega$ and $g_1,\dots,g_m:\omega^n \dashrightarrow \omega$ are partial recursive, then so is

$$h = \lambda s_1 \dots s_n.f(g_1(s_1,\dots,s_n),\dots,g_m(s_1,\dots,s_n)).$$

3. If $f:\omega \dashrightarrow \omega$ and $g:\omega^3 \dashrightarrow \omega$ are partial recursive, then so is $h:\omega^2 \dashrightarrow \omega$, where

$$h(s,0) = f(s),$$
$$h(s,t+1) = g(s,t,h(s,t)).$$

4. If $f:\omega^{n+1} \dashrightarrow \omega$ is partial recursive, then so is

$$h = \lambda s_1 \dots s_n.\mu t(f(s_1,\dots,s_n,t) = 0),$$

where $h(s_1,\dots,s_n)$ is the least t such that $f(s_1,\dots,s_n,t)=0$ and $f(s_1,\dots,s_n,r)\downarrow$ for all $r < t$, while $h(s_1,\dots,s_n)\uparrow$ if such t does not exist.

The respective operations in clauses 2–4 are called *composition*, *primitive recursion* and the *least number operator*. One gets the *primitive recursive functions* by skipping clause 4.

We shall modify the above definitions of primitive recursive and partial recursive functions to refer only to unary functions by making use of the fact that ω admits a primitive recursive pair coding, say $J = \lambda st.2^s(2t+1)-1$.

Take the class $\mathscr{F}_0 = \{\varphi/\varphi:\omega \dashrightarrow \omega\}$ and fix initial elements $\psi_0 = \lambda J(s,t).s$, $\psi_1 = \lambda J(s,t).t$, $\psi_2 = \lambda s.s$, $\psi_3 = \lambda s.0$, $\psi_4 = \lambda s.s+1$ and initial operations $\circ$,

$\Pi^*, Pr: \mathscr{F}_0^2 \to \mathscr{F}_0$ and $Ln: \mathscr{F}_0 \to \mathscr{F}_0$ such that

$$\varphi \circ \psi = \varphi\psi = \lambda s.\psi(\varphi(s)),$$
$$\Pi^*(\varphi, \psi) = \lambda s.J(\varphi(s), \psi(s)),$$
$$Pr(\varphi, \psi)(s) = \varphi(\psi_0(s)), \quad \text{if } \psi_1(s) = 0,$$
$$Pr(\varphi,\psi)(s) = \psi(J(\psi_0(s), J(\psi_1(s) - 1, Pr(\varphi,\psi)(J(\psi_0(s), \psi_1(s) - 1))))), \text{ if } \psi_1(s) > 0,$$
$$Ln(\varphi) = \lambda s.\mu t(\varphi(J(s,t)) = 0).$$

Then it can be shown that

$$\mathscr{U}_0 = cl(\psi_0, \ldots, \psi_4 / \circ, \Pi^*, Pr)$$

is the set of all unary primitive recursive functions, while

$$\mathscr{U} = cl(\psi_0, \ldots, \psi_4 / \circ, \Pi^*, Pr, Ln)$$

is the set of all unary partial recursive functions. Other initial elements and operations which also generate $\mathscr{U}$ will be suggested in chapter 3.

It is instructive to notice that two out of the above four operations have a least fixed point nature. Namely, let $\varphi \le \psi$ iff ψ is an extension of φ, i.e. iff $\varphi \subseteq \psi$. Making use of the function $\psi_5 = \lambda s.s \dot{-} 1 (\psi_5(0) = 0, \psi_5(s+1) = s)$ and the branching operation

$$(\chi \to \varphi, \psi)(s) = \begin{cases} \varphi(s), & \text{if } \chi(s) = 0, \\ \psi(s), & \text{if } \chi(s) > 0, \\ \uparrow, & \text{if } \chi(s)\uparrow, \end{cases}$$

it follows that $Pr(\varphi, \psi)$ is the least (and unique) fixed point of the mapping

$$\lambda\theta.(\psi_1 \to \psi_0\varphi, \Pi^*(\psi_0, \psi_1\psi_5)\Pi^*(\psi_0, \Pi^*(\psi_1, \theta))\psi)$$

while, as shown in Skordev [1980], $Ln(\varphi)$ equals $\Pi^*(\psi_2, \psi_3)$ multiplied by the least fixed point of the mapping

$$\lambda\theta.(\varphi \to \psi_1, \Pi^*(\psi_0, \psi_1\psi_4)\theta).$$

Of course, the class $\mathscr{F}_0$ is very special and will not be our only starting point. Here is a rather different one consisting of *monotonic functionals*:

$$\mathscr{F}_1 = \{\phi / \phi : \omega \times \mathscr{F}_0 \dashrightarrow \omega \ \&\ \phi \text{ is monotonic}\},$$

where ϕ is monotonic iff whenever $\phi(s, \varphi) = t$ and $\varphi \le \psi$, then $\phi(s, \psi) = t$. Alternatively, one may consider *operators* instead of functionals, the class

$$\mathscr{F}_2 = \{\Gamma / \Gamma : \mathscr{F}_0 \to \mathscr{F}_0 \ \&\ \Gamma \text{ is monotonic}\},$$

where Γ is monotonic iff whenever $\varphi \le \psi$, then $\Gamma(\varphi) \le \Gamma(\psi)$. To each functional $\phi \in \mathscr{F}_1$ there corresponds an operator $\lambda\theta.\lambda s.\phi(s, \theta) \in \mathscr{F}_2$ and vice versa, a functional $\lambda s\theta.\Gamma(\theta)(s) \in \mathscr{F}_1$ corresponds to each operator $\Gamma \in \mathscr{F}_2$.

Intending as we do to develop a general theory of effective computability, we make it our immediate task to choose abstract initial elements and operations to fit the above two classes as well as other similar ones. This is a key step which in effect determines the shape of the theory. Now, what kind of initial operations are we looking for? Initially, they should be few in number, simple and convenient. Secondly, they must be sufficiently non-trivial to support a rich general theory.

We shall extract four initial operations. The first one is fundamental and inherent in the concept of effective computability. (And so are the others, but this is less obvious at this stage.) The second operation is needed for technical reasons, while the last two are added to ensure that our collection of operations is in a certain sense complete. Let us be more specific. Letters $\varphi, \psi, \chi, \theta, \tau$ will stand for arbitrary members of $\mathscr{F}$.

Suppose first that an 'effective' *multiplication operation* $\circ: \mathscr{F}^2 \to \mathscr{F}$ and an element $I \in \mathscr{F}$ can be specified in such a way that a semigroup with a unit I is obtained. It is surely difficult to imagine operations more natural for $\mathscr{F}_0, \mathscr{F}_2$ than composition of functions or operators. Moreover composition is associative. So take $\varphi\psi = \lambda s.\psi(\varphi(s))$, $I = \lambda s.s$, respectively $\Gamma_1\Gamma_2 = \lambda\theta.\Gamma_1(\Gamma_2(\theta))$, $I = \lambda\theta.\theta$. Passing from operators to functionals, one gets $\phi\Psi = \lambda s\varphi.\phi(s, \lambda t.\Psi(t, \varphi))$, $I = \lambda s\varphi.\varphi(s)$ in $\mathscr{F}_1$.

Furthermore, we need a *pairing operation* for the members of $\mathscr{F}$, which is why we assume that there is an 'effective' operation $\Pi: \mathscr{F}^2 \to \mathscr{F}$ and 'effectively computable' elements $L, R \in \mathscr{F}$ such that $L(\varphi, \psi) = \varphi$, $R(\varphi, \psi) = \psi$ for all φ, ψ, writing for short (φ, ψ) for $\Pi(\varphi, \psi)$. To realize this in our two control classes, just take $(\varphi, \psi)(2s) = \varphi(s)$, $(\varphi, \psi)(2s+1) = \psi(s)$, $L = \lambda s.2s$, $R = \lambda s.2s+1$ in $\mathscr{F}_0$ and $(\phi, \Psi)(2s, \varphi) = \phi(s, \varphi)$, $(\phi, \Psi)(2s+1, \varphi) = \Psi(s, \varphi)$, $L = \lambda s\varphi.\varphi(2s)$, $R = \lambda s\varphi.\varphi(2s+1)$ in $\mathscr{F}_1$. (More details about the classes $\mathscr{F}_0$, $\mathscr{F}_1$ and their initial operations will be given in the next chapter.)

The operations $\circ$ and Π are naturally connected by the distributive law $(\varphi, \psi)\chi = (\varphi\chi, \psi\chi)$ which can be easily substantiated in both $\mathscr{F}_0$ and $\mathscr{F}_1$.

We obtain our remaining initial operations from multiplication and pairing by making use of least fixed points. Experience from Ordinary Recursion Theory suggests that it is reasonable to suppose that with respect to a certain partial ordering all 'effective' operations $\Gamma: \mathscr{F} \to \mathscr{F}$ have least fixed points which are 'effectively computable' elements. Assuming this and examining the least fixed points of some simple mappings constructed by means of $\circ$, Π, we show that two of them behave very nicely. The first is the least fixed point of $\lambda\theta.(\varphi L, \theta R)$, denoted by $\langle\varphi\rangle$, while the second is the least fixed point of $\lambda\theta.(I, \varphi\theta)$, denoted by $[\varphi]$. We call the operations $\lambda\varphi.\langle\varphi\rangle$, $\lambda\varphi.[\varphi]$ respectively *translation* and *iteration*, the names deriving from the syntax or semantics of the operations concerned.

A partial order has already been introduced in $\mathscr{F}_0$. As for $\mathscr{F}_1$, take $\phi \leq \Psi$ iff Ψ is an extension of ϕ. The existence of operations $\langle\ \ \rangle$, $[\ \ ]$ in both classes will be established in the next chapter.

The elements I, L, R and the operations $\circ, \Pi, \langle\ \ \rangle, [\ \ ]$ are all we need to start developing a general recursion theory. As a matter of fact, the above remarks reflect the very route by which the author arrived at the present approach in May 1979, with $\mathscr{F}$ being the class of all monotonic unary mappings over an arbitrary Skordev combinatory space.

It is interesting to see how $\langle\ \ \rangle$, $[\ \ ]$ look in $\mathscr{F}_0$ or $\mathscr{F}_1$; however this yields an intuition which is not so helpful in the general theory. In compensation, the following intuitive algebraic interpretation proves very suggestive.

Consider informally $\langle\varphi\rangle$ as $(\varphi\bar{0},(\varphi\bar{1},(\varphi\bar{2},\ldots)))$, where $\bar{n}=LR^n$, and $[\varphi]$ as $(I,\varphi(I,\varphi(I,\ldots)))$. Now, for instance, the equality $[\varphi]\psi=R[(\psi L,\varphi R)]$ is easy to deduce, but how does one guess what $[\varphi]\psi$ is equal to in advance? To see this, we agree informally to get

$$\begin{aligned}[\varphi]\psi &= (I,\varphi(I,\ldots))\psi=(\psi,\varphi(\psi,\ldots))=(\psi L,\varphi R)(I,(\psi L,\varphi R)(I,\ldots))\\ &=(\psi L,\varphi R)[(\psi L,\varphi R)]=R[(\psi L,\varphi R)].\end{aligned}$$

In another example,

$$\begin{aligned}\langle\varphi\rangle\langle\psi\rangle &= (\varphi\bar{0},(\varphi\bar{1},\ldots))\langle\psi\rangle=(\varphi\bar{0}\langle\psi\rangle,(\varphi\bar{1}\langle\psi\rangle,\ldots))\\ &=(\varphi\psi\bar{0},(\varphi\psi\bar{1},\ldots))=\langle\varphi\psi\rangle\end{aligned}$$

suggests that $\langle\varphi\rangle\langle\psi\rangle=\langle\varphi\psi\rangle$. Similarly in other cases, some of which are far more complicated.

Now that the above two equalities have been suggested, how does one prove them formally?

Take the equality $(I,\varphi[\varphi])=[\varphi]$. Multiplying by ψ, one gets $(\psi,\varphi[\varphi]\psi)=[\varphi]\psi$; hence $[\varphi]\psi$ is a fixed point of the mapping $\lambda\theta.(\psi,\varphi\theta)$. In order to prove the equality $[\varphi]\psi=R[(\psi L,\varphi R)]$, however, it is not sufficient to know that $[\varphi]$ is the least fixed point of $\lambda\theta.(I,\varphi\theta)$. One should be given a little more, namely that $[\varphi]\psi$ is the least fixed point of $\lambda\theta.(\psi,\varphi\theta)$. (The latter implies the former by taking $\psi=I$.) That granted, the proof is easy.

To begin with,

$$(I,(\psi L,\varphi R)(I,[\varphi]\psi))=(I,(\psi,\varphi[\varphi]\psi))=(I,(I,\varphi[\varphi])\psi)=(I,[\varphi]\psi);$$

hence $[(\psi L,\varphi R)]\leq(I,[\varphi]\psi)$. Multiplying on the left by R, we get $R[(\psi L,\varphi R)]\leq[\varphi]\psi$. On the other hand,

$$(\psi,\varphi R[(\psi L,\varphi R)])=(\psi L,\varphi R)[(\psi L,\varphi R)]=R[(\psi L,\varphi R)];$$

hence $[\varphi]\psi\leq R[(\psi L,\varphi R)]$. Therefore, $[\varphi]\psi=R[(\psi L,\varphi R)]$;

In order to prove $\langle\varphi\rangle\langle\psi\rangle=\langle\varphi\psi\rangle$ it suffices to be given that, whenever $R\chi=\chi\chi_1$, then $\langle\varphi\rangle\chi$ is the least fixed point of $\lambda\theta.(\varphi L\chi,\theta\chi_1)$. Actually, it follows that

$$(\varphi\psi L,\langle\varphi\rangle\langle\psi\rangle R)=(\varphi L\langle\psi\rangle,\langle\varphi\rangle R\langle\psi\rangle)=(\varphi L,\langle\varphi\rangle R)\langle\psi\rangle=\langle\varphi\rangle\langle\psi\rangle;$$

hence $\langle\varphi\psi\rangle\leq\langle\varphi\rangle\langle\psi\rangle$. On the other hand, $R\langle\psi\rangle=\langle\psi\rangle R$ (here $\chi=\langle\psi\rangle$, $\chi_1=R$) and

$$(\varphi L\langle\psi\rangle,\langle\varphi\psi\rangle R)=(\varphi\psi L,\langle\varphi\psi\rangle R)=\langle\varphi\psi\rangle;$$

hence $\langle\varphi\rangle\langle\psi\rangle\leq\langle\varphi\psi\rangle$. This completes the proof.

We collect all the necessary basic properties of $\langle\ \ \rangle$ and $[\ \ ]$ in an axiom called *μ-induction axiom*, *μ-axiom* for short. Several versions of this axiom are studied in this book. In essence, the properties assumed in the above two proofs compose the weakest axiom $\mu\mathrm{A}_0$:

$$\mu\mathrm{A}_0 \begin{cases} (\pounds) & (\varphi L, \langle \varphi \rangle R) \leq \langle \varphi \rangle, \\ & R\psi \leq \psi\psi_1 \,\&\, (\varphi L\psi, \tau\psi_1) \leq \tau \Rightarrow \langle \varphi \rangle \psi \leq \tau. \\ (\pounds\pounds) & (I, \varphi[\varphi]) \leq [\varphi], \\ & (\psi, \varphi\tau) \leq \tau \Rightarrow [\varphi]\psi \leq \tau. \end{cases}$$

This simple axiom turns out to engender almost all important statements of the general theory.

Thus we have designed an algebraic system intended to provide a framework for studying the concept of effective computability. We call this system an *operative space* since its elements are capable of *operating* in one way or another. An *operative space* (OS) is a 5-tuple $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ where $\mathscr{F}$ is a partially ordered semigroup with unit I, Π is a monotonic binary operation over $\mathscr{F}$ and L, R are distinct members of $\mathscr{F}$ such that $(\varphi, \psi)\chi = (\varphi\chi, \psi\chi)$, $L(\varphi, \psi) = \varphi$ and $R(\varphi, \psi) = \psi$. An *iterative operative space* (IOS) is an OS satisfying a certain μ-axiom, and we write IOS = OS + μA.

Our basic abstract notion of effective computability is that of *relative recursiveness* defined as follows. Given a subset $\mathscr{B}$ of $\mathscr{F}$, an element φ is *recursive in* $\mathscr{B}$ iff

$$\varphi \in cl(\{L, R\} \cup \mathscr{B} / \circ, \Pi, \langle \ \ \rangle, [\ \ ]),$$

i.e. iff φ can be constructed from L, R and members of $\mathscr{B}$ by means of $\circ, \Pi, \langle \ \ \rangle, [\ \ ]$. (The element I is not taken to be initial since $I = L[L]$.) This definition goes beyond what one might expect from the preceding discussion, for it introduces a notion of relative rather than absolute effective computability—an important step first made by Turing.

Starting with $\circ$ and Π, infinitely many 'effective' operations can be obtained via least fixed points. Such an operation is, say, the passage from φ, ψ to the least fixed point of $\lambda\theta.\varphi(I, \theta^2\psi)$. However, our choosing only two of these operations ($\langle \ \ \rangle$ and $[\ \ ]$) leads to no loss of generality since all the others can be expressed in terms of $\circ, \Pi, \langle \ \ \rangle, [\ \ ]$, as we will show later. 2nd reason!

Least fixed points have always played an important role in Recursion Theory; indeed the least fixed point phenomenon gave the theory its name. Least fixed point machinery has been widely exploited recently in Recursion Theory (e.g. Platek [1966], Moschovakis [1977, 1984], Feferman [1977], Kleene [1978], Skordev [1980]), in Inductive Definability Theory (Moschovakis [1974]), as well as in Computer Science (Scott [1971], de Bakker [1971], Manna [1974]). However, what is characteristic of the iterative spaces studied in the present book is the axiomatic treatment of recursion. Similar μ-induction principles are used in the Skordev combinatory spaces and the system of de Bakker and Scott. In fact, we shall see in chapter 26 that the axiom $\mu\mathrm{A}_0$ is a particular instance of Scott's μ-induction rule.

CHAPTER 3

Two examples

The axiomatic system of IOS outlined in the previous chapter has an enormous scope provided by its abundance of models. These are of two sorts; let us call them *first-order* and *higher order models*. The carrier of a first order model (such as $\mathscr{F}_0$) consists of function-like elements, while that of a higher order model (such as $\mathscr{F}_1$) consists of operator-like ones. As an illustration we present here two standard examples of IOS which will be studied in greater detail and in a more general form respectively in chapters 22, 28. It is convenient to use binary code, so that in this chapter $s0$ equals twice s (in particular, $00=0$), while $s1$ equals twice s plus one.

The first example corresponds to example 4 in Skordev [1980], chapter 2, and to example 1 in Skordev [1982].

Proposition 3.1 (Example 3.1). Let $\mathscr{F}=\{\varphi/\varphi:\omega \dashrightarrow \omega\}$ (the class $\mathscr{F}_0$ from the previous chapter), $\varphi\leq\psi$ iff $\varphi\subseteq\psi$, $\varphi\psi=\lambda s.\psi(\varphi(s))$, $(\varphi,\psi)(s0)=\varphi(s)$, $(\varphi,\psi)(s1)=\psi(s)$, $I=\lambda s.s$, $L=\lambda s.s0$ and $R=\lambda s.s1$. Then $\mathscr{S}=(\mathscr{F},I,\Pi,L,R)$ is an IOS.

Proof. Let us first check the axioms of OS. We have

$$\varphi(\psi\chi)(s)=(\psi\chi)(\varphi(s))=\chi(\psi(\varphi(s)))=\chi((\varphi\psi)(s))=(\varphi\psi)\chi(s);$$

hence $\varphi(\psi\chi)=(\varphi\psi)\chi$. We also get

$$\varphi I(s)=I(\varphi(s))=\varphi(s),\quad I\varphi(s)=\varphi(I(s))=\varphi(s);$$

hence $\varphi I=I\varphi=\varphi$. Therefore, $\mathscr{F}$ is a semigroup with unit I.

The monotonicity of $\circ,\Pi$ is immediate. The equalities

$$(\varphi,\psi)\chi(s0)=\chi(\varphi(s))=\varphi\chi(s)=(\varphi\chi,\psi\chi)(s0),$$
$$(\varphi,\psi)\chi(s1)=\chi(\psi(s))=\psi\chi(s)=(\varphi\chi,\psi\chi)(s1)$$

imply $(\varphi,\psi)\chi=(\varphi\chi,\psi\chi)$. Finally,

$$L(\varphi,\psi)(s)=(\varphi,\psi)(s0)=\varphi(s),$$
$$R(\varphi,\psi)(s)=(\varphi,\psi)(s1)=\psi(s)$$

give $L(\varphi,\psi)=\varphi$, $R(\varphi,\psi)=\psi$; hence $\mathscr{S}$ is an OS.

If $\{\varphi_n\}$ is an increasing sequence in $\mathscr{F}$, then $\varphi_0\subseteq\varphi_1\subseteq\ldots$, hence $\varphi=\bigcup_n\varphi_n\in\mathscr{F}$ and $\varphi=\sup_n\varphi_n$. Moreover, we have $\varphi\psi=\sup_n(\varphi_n\psi)$, $\psi\varphi=\sup_n\psi\varphi_n$

and $(\psi, \varphi) = \sup_n(\psi, \varphi_n)$ for all ψ. If $O = \lambda s.\uparrow$, then $O \leq \psi$ and $O\psi = O$ for all ψ. Using these properties, we are going to establish the validity of the axiom μA_0.

Take $\theta_0 = O$ and $\theta_{n+1} = (\varphi L, \theta_n R)$. The monotonicity of the mapping $\lambda\theta.(\varphi L, \theta R)$ implies by induction $\theta_n \leq \theta_{n+1}$ for all n. Define $\langle\varphi\rangle = \sup_n \theta_n$. Then

$$(\varphi L, \langle\varphi\rangle R) = \left(\varphi L, \left(\sup_n \theta_n\right) R\right) = \left(\varphi L, \sup_n(\theta_n R)\right) = \sup_n(\varphi L, \theta_n R)$$
$$= \sup_n \theta_{n+1} = \langle\varphi\rangle.$$

Suppose that $R\psi \leq \psi\psi_1$ and $(\varphi L\psi, \tau\psi_1) \leq \tau$. Then $\theta_0\psi = O\psi = O \leq \tau$ and whenever $\theta_n\psi \leq \tau$, then

$$\theta_{n+1}\psi = (\varphi L, \theta_n R)\psi = (\varphi L\psi, \theta_n R\psi) \leq (\varphi L\psi, \theta_n\psi\psi_1) \leq (\varphi L\psi, \tau\psi_1) \leq \tau,$$

hence

$$\langle\varphi\rangle\psi = \left(\sup_n \theta_n\right)\psi = \sup_n(\theta_n\psi) \leq \tau.$$

Therefore, (£) holds.

In the case of iteration take $\theta_0 = O$, $\theta_{n+1} = (I, \varphi\theta_n)$ and $[\varphi] = \sup_n \theta_n$. Then

$$(I, \varphi[\varphi]) = \left(I, \varphi \sup_n \theta_n\right) = \left(I, \sup_n \varphi\theta_n\right) = \sup_n(I, \varphi\theta_n) = \sup_n \theta_{n+1} = [\varphi].$$

Suppose that $(\psi, \varphi\tau) \leq \tau$. Then $\theta_0\psi = O \leq \tau$ and if $\theta_n\psi \leq \tau$, then

$$\theta_{n+1}\psi = (I, \varphi\theta_n)\psi = (\psi, \varphi\theta_n\psi) \leq (\psi, \varphi\tau) \leq \tau.$$

Therefore,

$$[\varphi]\psi = \left(\sup_n \theta_n\right)\psi = \sup_n(\theta_n\psi) \leq \tau$$

and (££) holds. The proof is complete.

The next statement establishes explicit characterizations of the operations $\langle\ \rangle$, $[\ ]$. We begin with two remarks.

Take $L_1 = L^{-1} = \lambda s0.s$ and $R_1 = R^{-1} = \lambda s1.s$. (In other words, $L_1 = (I, O)$, $R_1 = (O, I)$.) Then it follows that $LL_1 = RR_1 = I$, $LR_1 = RL_1 = O$ and $(\varphi, \psi) = L_1\varphi \cup R_1\psi$.

Each number s has a unique presentation as $s = t01^n$, 1^n standing for $1\ldots1$, n times

Proposition 3.2. Let $\mathscr{S}$ be the IOS of example 3.1. Then

$$\langle\varphi\rangle(s01^n) = \varphi(s)01^n,$$

$[\varphi](s) = t$ iff $\exists n r_0 \ldots r_n (r_0 = s \,\&\, \forall i < n(r_i \text{ is odd } \&\, r_{i+1} = \varphi(R_1(r_i))) \,\&\, r_n = t0)$.

Proof. Multiplying the equality $\langle\varphi\rangle = (\varphi L, \langle\varphi\rangle R)$ on the left by $\bar{n}(= LR^n)$ one gets $\bar{n}\langle\varphi\rangle = \varphi\bar{n}$ for all n. Therefore,

$$\langle\varphi\rangle(s01^n) = \bar{n}\langle\varphi\rangle(s) = \varphi\bar{n}(s) = \varphi(s)01^n.$$

Take $\theta_0 = O$ and $\theta_{n+1} = (I, \varphi\theta_n)$ as in the proof of 3.1. Obviously, $\theta_0 = \bigcup_{i<0}(R_1\varphi)^i L_1$. If $\theta_n = \bigcup_{i<n}(R_1\varphi)^i L_1$, then

$$\theta_{n+1} = L_1 \cup R_1\varphi\theta_n = \bigcup_{i<n+1}(R_1\varphi)^i L_1,$$

hence $[\varphi] = \sup_n \theta_n = \bigcup_n (R_1\varphi)^n L_1$. If there exists an n such that $(R_1\varphi)^n L_1(s)\downarrow$, then it is unique (being the least m for which $(R_1\varphi)^m(s)$ is even), hence $[\varphi](s) = (R_1\varphi)^n L_1(s)$. Otherwise $[\varphi](s)\uparrow$. This yields the desired characterization of $[\quad]$, which completes the proof.

It follows from 3.2 that translation is a *primitive recursive operation*, which means that $\langle\varphi\rangle$ can be constructed by clauses 1–3 stated in the previous chapter with φ added to the initial functions in clause 1, employing one and the same construction for all φ. In other words, $\langle\varphi\rangle$ is uniformly primitive recursive in φ.

As for iteration, 3.2 shows that it is a *μ-recursive operation*. That is, for all φ the function $[\varphi]$ can be constructed by clauses 1–4 with φ allowed in clause 1, using one and the same construction for all φ; in other words, $[\varphi]$ is uniformly μ-recursive in φ. (Why not say 'partial recursive in φ'? Because it differs from 'μ-recursive in φ'; see the comments to exercise 8.2***.)

On the other hand, it will be shown in chapter 22 that all μ-recursive operations over $\mathscr{F}$ can be expressed by means of $\circ, \Pi\,\langle\quad\rangle, [\quad]$. In particular, the unary partial recursive functions are exactly the members of $\mathscr{F}$ IOS-recursive in Z, where $Z(0) = 0$ and $Z(s) = s1$ otherwise.

The following example is based on the class $\mathscr{F}_1$ of chapter 2.

Proposition 3.3 (Example 3.2). Let $\mathscr{F}_0 = \{\varphi/\varphi{:}\omega \dashrightarrow \omega\}$. Take

$$\begin{aligned}
&\mathscr{F} = \{\phi/\phi{:}\omega \times \mathscr{F}_0 \dashrightarrow \omega \,\&\, \phi \text{ is monotonic}\},\\
&\phi \le \Psi \text{ iff } \Psi \text{ is an extension of } \phi,\\
&\phi\Psi = \lambda s\varphi.\phi(s, \lambda t.\Psi(t, \varphi)),\\
&(\phi, \Psi)(s0, \varphi) = \phi(s, \varphi),\\
&(\phi, \Psi)(s1, \varphi) = \Psi(s, \varphi),\\
&I = \lambda s\varphi.\varphi(s), L = \lambda s\varphi.\varphi(s0), R = \lambda s\varphi.\varphi(s1)
\end{aligned}$$

Then $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is an IOS.

Proof. Verifying the axioms of OS is left to the reader as an easy exercise. We shall establish the validity of μA_0 by making use of the fact that whenever $\{\phi_\eta\}_{\eta<\xi}$ is an increasing transfinite sequence in $\mathscr{F}$, then $\phi = \bigcup_{\eta<\xi}\phi_\eta \in \mathscr{F}$, $\phi = \sup_{\eta<\xi}\phi_\eta$, $\phi\Psi = \sup_{\eta<\xi}(\phi_\eta\Psi)$ and $(\Psi, \phi) = \sup_{\eta<\xi}(\Psi, \phi_\eta)$ for all Ψ.

Translation is treated just as in 3.1, defining $\langle\phi\rangle$ as $\sup_n \Theta_n$, where $\Theta_0 = O = \lambda s\varphi.\uparrow$ and $\Theta_{n+1} = (\phi L, \Theta_n R)$.

In the case of iteration, however, one has to consider transfinite sequences, for the equality $\phi \sup_n \Theta_n = \sup_n \phi\Theta_n$ may fail for discontinuous ϕ.

Suppose that elements Θ_η have been constructed such that $\Theta_\eta = \sup_{\zeta<\eta}(I, \phi\Theta_\zeta)$ for all $\eta < \xi$. Then the sequence $\{\Theta_\eta\}_{\eta<\xi}$ is obviously increasing, hence so is $\{(I, \phi\Theta_\eta)\}_{\eta<\xi}$ and the element $\Theta_\xi = \sup_{\eta<\xi}(I, \phi\Theta_\eta)$ exists. In this way one gets an increasing transfinite sequence $\{\Theta_\xi\}$. Because of cardinality reasons it cannot consist of distinct members, hence there is a ζ such that

$\Theta_{\zeta+1}=\Theta_\zeta$; any cardinal $\zeta>\mathrm{Card}(\mathscr{F})$ will do. Now put $[\phi]=\Theta_\zeta$, then

$$(I,\phi[\phi])=(I,\phi\Theta_\zeta)=\Theta_{\zeta+1}=[\phi].$$

It remains to show that $(\Psi,\phi\Sigma)\le\Sigma$, implies $[\phi]\Psi\le\Sigma$. Supposing $\Theta_\eta\Psi\le\Sigma$ for all $\eta<\xi$, one gets

$$\Theta_\xi\Psi=\left(\sup_{\eta<\xi}(I,\phi\Theta_\eta)\right)\Psi=\sup_{\eta<\xi}((I,\phi\Theta_\eta)\Psi)=\sup_{\eta<\xi}(\Psi,\phi\Theta_\eta\Psi)$$
$$\le(\Psi,\phi\Sigma)\le\Sigma.$$

Therefore, $\Theta_\xi\Psi\le\Sigma$ for all ξ, hence $[\phi]\Psi\le\Sigma$. The proof is complete.

The following statement characterizes the operations $\langle\ \ \rangle$, $[\ \ ]$ of example 3.2 explicitly. Being by definition a particular instance of the least fixed point operator over $\mathscr{F}$, $[\phi]=\mu\Theta.(I,\phi\Theta)$, iteration turns out to be at the same time the general least fixed point operator over $\mathscr{F}_0$.

Proposition 3.4. Let $\mathscr{S}$ be the space of example 3.2. Then

$$\langle\phi\rangle(s01^n,\varphi)=\phi(s,\lambda t.\varphi(t01^n)),$$

while for all φ the function $\sigma=\lambda s.[\phi](s,\varphi)$ is the least solution of the equality

$$\theta(s\varepsilon)=\begin{cases}\varphi(s), & \text{if } \varepsilon=0,\\ \phi(s,\theta), & \text{if } \varepsilon=1,\end{cases}\tag{1}$$

or $\sigma=\mu\theta.(\varphi,\lambda s.\phi(s,\theta))$ in terms of example 3.1.

Proof. The equality $\bar{n}\langle\phi\rangle=\phi\bar{n}$ implies

$$\langle\phi\rangle(s01^n,\varphi)=\bar{n}\langle\phi\rangle(s,\varphi)=\phi\bar{n}(s,\varphi)=\phi(s,\lambda t.\bar{n}(t,\varphi))=\phi(s,\lambda t.\varphi(t01^n)).$$

As for iteration, notice first that for all $\psi\in\mathscr{F}_0$ the functional $\check{\psi}=\lambda s\varphi.\psi(s)$ is in $\mathscr{F}$ and $\check{\sigma}=[\phi]\check{\varphi}$.

Multiplying the equality $[\phi]=(I,\phi[\phi])$ by $\check{\varphi}$, one gets $\check{\sigma}=(\check{\varphi},\phi\check{\sigma})$, hence σ satisfies the equality (1) which can also be written as $\check{\theta}=(\check{\varphi},\phi\check{\theta})$. If τ is another solution to (1), then $\check{\tau}=(\check{\varphi},\phi\check{\tau})$ implies $[\phi]\check{\varphi}\le\check{\tau}$ by (££). Therefore, $\check{\sigma}\le\check{\tau}$, hence $\sigma\le\tau$, which completes the proof.

Recursion in specific functionals. It follows from 12.38*, 22.4 that $\Gamma:\mathscr{F}_0\to\mathscr{F}_0$ is a μ-recursive operation iff $\lambda s\theta.\Gamma(\theta)(s)$ is an element of $\mathscr{F}$ recursive in $\check{Z}, Id, Ml$, where $Id=\lambda s\theta.s$, $Ml=\lambda s\theta.\theta(\theta(s0)1)$.

A function $\varphi\in\mathscr{F}_0$ can be shown to have a Π^1_1 graph iff the element $\check{\varphi}$ of $\mathscr{F}$ is recursive in $\check{Z}, Id, Ml, E^\#$ (Kleene [1959], Kechris and Moschovakis [1977]), where $E^\#(s,\theta)=0$, if $\exists t(\theta(s01^t)=0)$, $E^\#(s,\theta)=1$, if $\forall t(\theta(s01^t)>0)$, and $E^\#(s,\theta)\uparrow$ otherwise. If φ is total, then one can replace 'Π^1_1' by 'Δ^1_1, i.e. hyperarithmetical'.