PART F

Connection with other theories

CHAPTER 26

Mathematical theory of programs

This chapter clarifies the connection between IOS and certain related algebraic systems of Computer Science, including the programming spaces of Skordev [1982] and the system of de Bakker and Scott introduced in the unpublished work de Bakker and Scott [1969]. Results of the general IOS-theory are transferred to appropriate extensions of the latter system.

The programming spaces of Skordev are easy to describe. Ignoring some minor technical differences, they are nothing else but OS augmented with iteration satisfying (££), i.e. IOS without translation. A version of this system is studied by Georgieva [1980].

The system of de Bakker and Scott is a formal one, hence a comparable formal system for IOS is needed. In fact we are going to introduce two such systems, $\mathscr{S}_0$ and $\mathscr{S}_1$.

System $\mathscr{S}_0$. Its language $\mathscr{L}_0$ has constants I, L, R and variables denoted by $\theta, \theta_1, \theta_2, \ldots$. Terms are constructed by the operations $\varphi\psi$, (φ, ψ), $\langle\varphi\rangle$ and $[\varphi]$. Formulas are equalities $\varphi = \psi$.

The axioms and rules of $\mathscr{S}_0$ are those of equality and substitution plus the following nonlogical axioms.

$$(\theta_1\theta_2)\theta_3 = \theta_1(\theta_2\theta_3), \quad \theta I = \theta, \quad I\theta = \theta,$$
$$(\theta_1, \theta_2)\theta = (\theta_1\theta, \theta_2\theta),$$
$$L(\theta_1, \theta_2) = \theta_1, \ R(\theta_1, \theta_2) = \theta_2,$$
$$(\theta L, \langle\theta\rangle R) = \langle\theta\rangle, \quad (I, \theta[\theta]) = [\theta].$$

What is interesting about this simple formal system is that it provides for the weak representability of all the partial recursive functions, which implies in turn that $\mathscr{S}_0$ is essentially undecidable, That is, all consistent extensions of $\mathscr{S}_0$ are undecidable. Consistency here means that $\overline{0} = O$ is not deducible, hence $\overline{0} = \overline{1}$ is not deducible. (As usual, $\bar{n}$, O stand for LR^n, $R[R]$.) The system $\mathscr{S}_0$ itself is consistent since every IOS is a model of it.

Proposition 26.1 (Undecidability Theorem). $\mathscr{S}_0$ is essentially undecidable.

Proof. Following an argument of Skordev [1980], take a partial recursive function $f:\omega \dashrightarrow \{0,1\}$ which has no general recursive extension. (For instance, if σ is a unary partial recursive function universal for all such functions and g is a supposed general recursive extension of $f = \lambda s.1 \dot{-} \bar{s}\sigma(s)$, then $g = \bar{n}\sigma$ for a certain n, hence $g(n) = 1 \dot{-} g(n)$, which is not the case.) The proof of 8.2 implies that there is an $\mathscr{L}_0$-term φ such that whenever $f(s) = t$, then $\bar{s}\varphi = \bar{t}$ is deducible in $\mathscr{o}_0$, written $\vdash_{\mathscr{o}_0} \bar{s}\varphi = \bar{t}$.

Suppose that $\mathscr{T}$ is a decidable consistent extension of $\mathscr{o}_0$. Then the function

$$g(s) = \begin{cases} 0, & \text{if} \vdash_{\mathscr{T}} \bar{s}\varphi = \bar{0}, \\ 1, & \text{if} \nvdash_{\mathscr{T}} \bar{s}\varphi = \bar{0} \end{cases}$$

is general recursive. If $f(s) = 0$, then $\vdash_{\mathscr{o}_0} \bar{s}\varphi = \bar{0}$; hence $\vdash_{\mathscr{T}} \bar{s}\varphi = \bar{0}$ and $g(s) = 0$. If $f(s) = 1$, then similarly $\vdash_{\mathscr{T}} \bar{s}\varphi = \bar{1}$; hence $\nvdash_{\mathscr{T}} \bar{s}\varphi = \bar{0}$ and $g(s) = 1$. Therefore g is a general recursive extension of f, which is not the case. The proof is complete.

The system $\mathscr{o}_1, \mathscr{o}_3, \mathscr{o}_4$ studied in this chapter are consistent extensions of $\mathscr{o}_0$, hence undecidable. One may also want to know whether they are incomplete with respect to a certain standard model or class of models, say the class of models consisting of partial single-valued (or multiple-valued) functions. The answer is positive; all consistent recursively axiomatizable extensions of $\mathscr{o}_0$ are incomplete with respect to a wide range of natural classes of models.

A structure $\mathscr{S}$ the language of which extends $\mathscr{L}_0$ is said to *have the representation property* iff whenever f is a partial recursive function, φ is a variable-free $\mathscr{L}_0$-term weakly representing f by the proof of 8.2 and $f(s)\uparrow$, then $\models_{\mathscr{S}} \bar{s}\varphi = O$, i.e. $\bar{s}\varphi = O$ is valid in $\mathscr{S}$. For instance, all μA_3-iterative OS $\mathscr{S}$ have this property.

The following statement is called the Incompleteness Theorem.

Proposition 26.2. Let $\mathscr{T}$ be a recursively axiomatizable extension of $\mathscr{o}_0$ and let $\mathscr{K}$ be a class of models of $\mathscr{T}$ which have the representation property, assuming $\not\models_{\mathscr{K}} \bar{0} = O$. Then there are $\mathscr{L}_0$-terms ρ, σ such that $\models_{\mathscr{K}} \rho = \sigma$ but $\nvdash_{\mathscr{T}} \rho = \sigma$.

Proof. The idea is again due to Skordev. Take f as in the previous proof, $g(s) = 0$, if $f(s) = 0$, and $g(s)\uparrow$ otherwise; then take a $\mathscr{L}_0$-term ψ weakly representing g by the proof of 8.2.

The function $h(s) = 1$, if $\vdash_{\mathscr{T}} \bar{s}\psi = O$, and $h(s)\uparrow$ otherwise, is partial recursive by the recursive axiomatizability of $\mathscr{T}$. By way of contradiction, suppose that $\models_{\mathscr{K}} \bar{s}\psi = O$ implies $h(s) = 1$ for all s.

If $f(s) = 0$, then $g(s) = 0$; hence $\vdash_{\mathscr{o}_0} \bar{s}\psi = \bar{0}$, which implies $\models_{\mathscr{K}} \bar{s}\psi = \bar{0}$ and $h(s)\uparrow$. Otherwise $g(s)\uparrow$ gives $\models_{\mathscr{K}} \bar{s}\psi = O$ by the representation property, hence $h(s) = 1$. We get $f \leq g \cup h$, which is not the case since $U(g,h)$ is a general recursive function by 22.4. Therefore, there is an s such that $\models_{\mathscr{K}} \bar{s}\psi = O$ but $\nvdash_{\mathscr{T}} \bar{s}\psi = O$. The proof is complete.

In particular, the role of $\mathscr{T}$ can be played by $\mathscr{o}_0$ itself. The latter system is however complete with respect to the canonical model consisting of all $\mathscr{L}_0$-terms factorized by assuming $\varphi=\psi$ if $\vdash_{\mathscr{S}_0}\varphi=\psi$.

A modification of the Incompleteness Theorem ensures that all consistent extensions of $\mathscr{S}_0$ which provide representability of the partial recursive functions are not recursively axiomatizable.

Proposition 26.3. Let $\mathscr{T}_1$ be a consistent extension of $\mathscr{S}_0$ such that whenever f is a partial recursive function, φ is a $\mathscr{L}_0$-term corresponding to f by the proof of 8.2 and $f(s)\uparrow$, then $\vdash_{\mathscr{T}_1}\bar{s}\varphi=O$. Then $\mathscr{T}_1$ is not recursively axiomatizable.

Proof. Suppose that $\mathscr{T}$ is a recursively axiomatized extension of $\mathscr{S}_0$ and $\mathscr{T}_1$ is an extension of $\mathscr{T}$. Repeating the proof of 26.2 with $\vdash_{\mathscr{T}_1}$ substituted for $\vDash_{\mathscr{K}}$, one gets $\mathscr{L}_0$-terms ρ, σ such that $\vdash_{\mathscr{T}_1}\rho=\sigma$ but $\nvdash_{\mathscr{T}}\rho=\sigma$, hence $\mathscr{T}$, $\mathscr{T}_1$ are not equivalent. This completes the proof.

It follows from 26.3 that the theory of μA_3-iterative OS is equivalent to no recursively axiomatized first order theory.

System $\mathscr{S}_1$. The language $\mathscr{L}_0$ is extended to a wider one $\mathscr{L}_1$ as follows. Terms are now constructed by the operations $\varphi\psi$, (φ,ψ) and the variable binding *μ-operation* $(\mu\theta.\varphi)$, while $\langle\varphi\rangle$, $[\varphi]$ stand respectively for $\mu\theta.(\varphi L,\theta R)$, $\mu\theta.(I,\varphi\theta)$ for a certain θ not free in φ, say the first such θ. Atomic formulas ϕ,Ψ are equalities $\varphi=\psi$ and inequalities $\varphi\le\psi$. Formulas are implications $\vec{\phi}\to\vec{\Psi}$, where $\vec{\phi},\vec{\Psi}$ are conjunctions of atomic formulas written as finite lists. If $\vec{\phi}$ is an empty list, then we write simply $\vec{\Psi}$ for $\to\vec{\Psi}$.

The logical axioms and rules of $\mathscr{S}_1$ are those of equality and substitution plus

$$\phi_1,\dots,\phi_n\to\phi_i,\quad \text{all } 1\le i\le n,$$

$$\frac{\vec{\phi}\to\Psi_i,\ 1\le i\le n}{\vec{\phi}\to\Psi_1,\dots,\Psi_n},$$

$$\frac{\vec{\phi}\to\vec{\Psi},\ \vec{\Psi}\to\vec{X}}{\vec{\phi}\to\vec{X}}.$$

The nonlogical axioms are as follows.

$$\begin{aligned}
&\theta_1\le\theta_2,\theta_3\le\theta_4\to\theta_1\theta_3\le\theta_2\theta_4,\\
&(\theta_1\theta_2)\theta_3=\theta_1(\theta_2\theta_3),\ \theta I=\theta,\ I\theta=\theta,\\
&\theta_1\le\theta_2,\theta_3\le\theta_4\to(\theta_1,\theta_3)\le(\theta_2,\theta_4),\\
&(\theta_1,\theta_2)\theta=(\theta_1\theta,\theta_2\theta),\\
&L(\theta_1,\theta_2)=\theta_1,\quad R(\theta_1,\theta_2)=\theta_2,\\
&\varphi(\mu\theta.\varphi/\theta)\le\mu\theta.\varphi,
\end{aligned}$$

where $\varphi(\psi/\theta)$ is φ with ψ substituted for the free occurrences of θ. Finally, $\mathscr{S}_1$ has a nonlogical rule of inference,

$$(\mu)\qquad \frac{\vec{\phi},\Psi\to\Psi(\varphi/\theta)}{\vec{\phi}\to\Psi(\mu\theta.\varphi/\theta)},$$

where Ψ is either $\theta\chi \leq \tau$ or $\langle\theta\rangle \leq \langle I\rangle\tau$, and θ is not free in $\vec{\phi}, \chi, \tau$.

All the theorems of the formal system $\mathscr{s}_1$ are obviously theorems of the theory of μA_1-iterative OS. Conversely, a number of the statements proved in chapters 4–9 are theorems or metatheorems of $\mathscr{s}_1$. Since $\mathscr{s}_1$ has no negation, the role of $\vec{\phi} \to \neg\vec{\Psi}$ is played by $\vec{\phi}, \vec{\Psi} \to L = R$.

System $\mathscr{s}_2$. The system of de Bakker and Scott. Its language $\mathscr{L}_2$ has constants I, O, function variables $\theta, \theta_1, \theta_2, \ldots$ and predicate variables P, P_1, $P_2, \ldots$. Terms are constructed by the operations $\varphi\psi$, $(P \to \varphi, \psi)$ and $\mu\theta.\varphi$. Formulas are introduced as in $\mathscr{L}_1$ and the logical axioms and rules of $\mathscr{s}_2$ are those of $\mathscr{s}_1$.

The nonlogical axioms include the axioms of a partially ordered semigroup with unit I and zero O, and the following axioms for conditionals due to McCarthy [1963].

$$(P \to I, I) \leq I,$$
$$\theta_1 \leq \theta_2, \theta_3 \leq \theta_4 \to (P \to \theta_1, \theta_3) \leq (P \to \theta_2, \theta_4),$$
$$(P \to (P \to \theta_1, \theta_2), \theta_3) = (P \to \theta_1, \theta_3),$$
$$(P \to \theta_1, (P \to \theta_2, \theta_3)) = (P \to \theta_1, \theta_3),$$
$$(P_1 \to (P_2 \to \theta_1, \theta_2), (P_2 \to \theta_3, \theta_4)) = (P_2 \to (P_1 \to \theta_1, \theta_3), (P_1 \to \theta_2, \theta_4)),$$
$$(P \to \theta_1, \theta_2)\theta = (P \to \theta_1\theta, \theta_2\theta).$$

The μ-operation satisfies the axiom

$$\varphi(\mu\theta.\varphi/\theta) \leq \mu\theta.\varphi$$

and Scott's μ-induction rule

$$\frac{\vec{\phi} \to \vec{\Psi}(O/\theta),\ \vec{\phi}, \vec{\Psi} \to \vec{\Psi}(\varphi/\theta)}{\vec{\phi} \to \vec{\Psi}(\mu\theta.\varphi/\theta)},$$

where θ is not free in $\vec{\phi}$.

This formal system has been extensively studied by de Bakker [1971], its relevance to Computer Science residing in the fact that $\mathscr{L}_2$-terms provide yet another presentation of unary recursive program schemes; more details on this connection will be given in the exercises. The unary (recursion-free) schemes correspond to the so called regular terms introduced as follows, $\mathrm{Fr}(\varphi)$ standing for the set of all free variables of φ.

Each variable θ is a term *regular in* θ. If $\theta \notin \mathrm{Fr}(\varphi)$ and ψ is regular in θ, then both φ and $\varphi\psi$ are regular in θ. If φ, ψ are regular in θ, then so is $(P \to \varphi, \psi)$. If φ is regular in θ, θ_1, then $\mu\theta.\varphi$ is regular in θ_1.

All function constants and variables are *regular terms*. If φ, ψ are regular, then so are $\varphi\psi$ and $(P \to \varphi, \psi)$. If φ is regular in θ and regular, then $\mu\theta.\varphi$ is regular.

In order to apply the IOS-theory to the theory of program schemes, we design a formal system to extend both $\mathscr{s}_1$ and $\mathscr{s}_2$.

System $\mathscr{s}_3$. Its language $\mathscr{L}_3$ is obtained from $\mathscr{L}_2$ by adding new function constants L, R, K and a predicate constant Ev. (One may also add a predicate constant Ze as in chapter 23.) The axioms and rules of $\mathscr{I}_3$ are those of $\mathscr{I}_2$ plus

$$L(Ev \to K\theta_1, K\theta_2) = \theta_1, \quad R(Ev \to K\theta_1, K\theta_2) = \theta_2.$$

The introduction of L, R, K, Ev corresponds to the additional assumptions of Böhm and Jacopini [1966] and de Bakker [1971], p. 46. In other words, we have introduced a counter, hence the $\mathscr{L}_3$-terms are unary recursive schemes with one counter, while the regular $\mathscr{L}_3$-terms are unary schemes with one counter.

Proposition 26.4. $\mathscr{I}_3$ is an extension of $\mathscr{I}_1$.

Proof. Writing (φ, ψ) for $(Ev \to K\varphi, K\psi)$, the language $\mathscr{L}_3$ is an extension of $\mathscr{L}_1$ and the axioms of $\mathscr{I}_1$ are deducible in $\mathscr{I}_3$, while the inference rules of $\mathscr{I}_1$ other than (μ) are rules of $\mathscr{I}_3$ as well. The rule (μ) however turns out to be a particular instance of Scott's rule. In order to prove this it suffices to show that $\vdash_{\mathscr{I}_3} \Psi(O/\theta)$ for all Ψ allowed in (μ). The assertion $\vdash_{\mathscr{I}_3} O\chi \leq \tau$ is immediate. If Ψ is $\langle \theta \rangle \leq \langle I \rangle \tau$, then we get $\vdash_{\mathscr{I}_3} \langle O \rangle \leq \langle I \rangle O$ by Scott's rule, hence $\vdash_{\mathscr{I}_3} \langle O \rangle \leq \langle I \rangle \tau$. This completes the proof.

The following technical statement is easily proved by using $\vdash_{\mathscr{I}_3} (P \to \varphi, \psi) = \tilde{P}(\varphi, \psi)$, where $\tilde{P}$ stands for $(P \to L, R)$.

Proposition 26.5. For every $\mathscr{L}_3$-term φ with predicate variables $P_1, \ldots, P_m$ there is an $\mathscr{L}_1$-term ψ and function variables $\theta_1, \ldots, \theta_{m+2}$ such that $\vdash_{\mathscr{I}_3} \varphi = \psi(K, \widetilde{Ev}, \tilde{P}_1, \ldots, \tilde{P}_m/\theta_1, \ldots, \theta_{m+2})$.

We introduce certain varieties of $\mathscr{L}_3$-terms in accordance with the notions of chapter 7, 'recursive' replaced by 'canonical' to avoid confusion. A term is *primitive* iff I does not occur in it and the μ-operation occurs as $\langle \quad \rangle$ only. A term is *canonical* (respectively, *prime canonical*) iff the μ-operation occurs in it only as $\langle \quad \rangle, [\quad]$ (as $[\quad]$).

Notice that all prime canonical terms are regular, which is not the case for the primitive or canonical terms. The proof of 23.1 shows that the prime canonical terms are structured schemes with one counter, while the canonical ones are structured schemes with one counter and a very special form of recursion compensating for the lack of a second counter.

Propositions 26.4, 26.5 make it possible to restate for $\mathscr{I}_3$ some basic IOS-results without having to repeat their proofs once again. To begin with, the following Recursion Elimination and Structurization Theorem corresponds to 9.15*.

Proposition 26.6. For every term φ there is a canonical term ψ such that $\vdash_{\mathscr{I}_3} \varphi = \psi$.

By 26.6, the following Normal Form Theorem and Enumeration Theorem correspond respectively to 9.4, 9.19.

Proposition 26.7. For every term φ and all $P_1, \ldots, P_m$, $\theta_1, \ldots, \theta_n$ there is a primitive term ψ such that $P_1,\ldots,P_m,\theta_1,\ldots,\theta_n \notin \mathrm{Fr}(\psi)$ and

$$\vdash_{\mathscr{I}_3} \varphi = \bar{\mathrm{I}}[\psi(I,\langle \tilde{P}_1\rangle,\ldots,\langle \tilde{P}_m\rangle,\langle\theta_1\rangle,\ldots,\langle\theta_n\rangle)].$$

In particular, $\vdash_{\mathscr{I}_3} \varphi = \bar{\mathrm{I}}[\psi]$ whenever $m=n=0$.

Proposition 26.8. For all $P_1,\ldots,P_m,\theta_1,\ldots,\theta_n$ there is a canonical term σ such that whenever $\mathrm{Fr}(\varphi)\subseteq\{P_1,\ldots,P_m,\theta_1,\ldots,\theta_n\}$, then $\vdash_{\mathscr{I}_3}\varphi=\bar{k}\sigma$ for a certain $\bar{k}$.

The following statement shows that all unary schemes with one counter can be structurized—a well known result of Böhm and Jacopini [1966] reaffirmed by de Bakker [1971], Skordev [1982] and Georgieva [1980].

Proposition 26.9. Let φ be regular in $\theta_1,\ldots,\theta_n$ and regular. Then there is a prime canonical ψ such that $\theta_1,\ldots,\theta_n\notin\mathrm{Fr}(\psi)$ and $\vdash_{\mathscr{I}_3}\varphi=\psi(I,\theta_1,\ldots,\theta_n)$. In particular, if φ is regular, then $\vdash_{\mathscr{I}_3}\varphi=\psi$ for a certain prime canonical ψ.

Proof. By induction on the construction of φ; the induction step for the μ-operation follows.

Let $\vdash_{\mathscr{I}_3}\varphi_1=\psi(I,\theta_1,\ldots,\theta_n)$ and $\varphi=\mu\theta_n.\varphi_1$. Writing ρ for $\psi(\bar{0}L,\ldots,\overline{n-2}L,R^{n-1}L,R)$, it follows that $\vdash_{\mathscr{I}_3}\varphi_1=\rho((I,\theta_1,\ldots,\theta_{n-1}),\theta_n)$; hence $\vdash_{\mathscr{I}_3}\varphi=R[\rho](I,\theta_1,\ldots,\theta_{n-1})$ by 6.11. The proof is complete.

Combining 26.9 and 9.8, one gets a Normal Form Theorem for regular terms.

Proposition 26.10. For every regular term φ, every θ and n not less than the number of the free occurrences of θ in φ there is a term ψ free of I and μ-operation such that $\theta\notin\mathrm{Fr}(\psi)$ and

$$\vdash_{\mathscr{I}_3}\varphi=\bar{\mathrm{I}}[\psi(I,\theta\bar{4},\ldots,\theta\overline{n+3})].$$

The proof of 23.1 suggests that [ ] is equivalent to looping and indeed this can be formally expressed by the equalities $\mu\theta.(P\to I,\varphi\theta)=\tilde{P}[\varphi\tilde{P}]$, $[\varphi]=(\mu\theta.(Ev\to I,K\varphi\theta))K$, θ not free in φ, which are easily deducible in $\mathscr{I}_3$.

As far as the construct $\langle\ \rangle$ is concerned, we know that its implementation requires an extra counter. For this purpose we introduce the following extension of the formal system $\mathscr{I}_3$.

System $\mathscr{I}_4$. Its language $\mathscr{L}_4$ extends $\mathscr{L}_3$ by adding new function constants L_2,R_2,K_2, a new predicate constant Ev_2 and, eventually, another predicate constants Ze_2. The new axioms are

$$\begin{aligned}
&LL_2=L_2L,\ LR_2=R_2L,\ LK_2=K_2L,\\
&RL_2=L_2R,\ RR_2=R_2R,\ RK_2=K_2R,\\
&KL_2=L_2K,\ KR_2=R_2K,\\
&\widetilde{Ev}L_2=L_2\widetilde{Ev},\ \widetilde{Ev}R_2=R_2\widetilde{Ev},\\
&L_2\widetilde{Ev}_2K_2=L,\ R_2\widetilde{Ev}_2K_2=R,\\
&L\widetilde{Ev}_2=(Ev_2\to L^2,LR),\ R\widetilde{Ev}_2=(Ev_2\to RL,R^2).
\end{aligned}$$

$\mathscr{s}_4$ extends $\mathscr{s}_3$ in such a way that 26.6–26.10 hold for $\mathscr{L}_4$-terms and $\mathscr{s}_4$-deducibility as well. Thus we come to the following Recursion Elimination and Structurization Theorem.

Proposition 26.11. Let $\mathrm{Fr}(\varphi) = \{P_1, \dots, P_m, \theta_1, \dots, \theta_n\}$. Then there is a prime canonical term ψ such that

$$\vdash_{\mathscr{s}_3} \tilde{P}_i L_2 = L_2 \tilde{P}_i, \tilde{P}_i R_2 = R_2 \tilde{P}_i, 1 \le i \le m, \theta_j L_2 = L_2 \theta_j,$$
$$\theta_j R_2 = R_2 \theta_j, 1 \le j \le n \to \varphi = \psi.$$

In view of 26.6 φ may be assumed canonical, so the proof formalizes that of the implication (3)$\Rightarrow$(2) of 23.1. The new axioms of $\mathscr{s}_4$ ensure that $W = \widetilde{Ev}_2 K_2$, $W_1 = L_2$, $W_2 = R_2$ satisfy the assumptions of 21.10. In particular, $\langle I \rangle L_2 = L_2 \langle I \rangle$ and $\langle I \rangle R_2 = R_2 \langle I \rangle$ follow from Scott's rule.

In terminology of chapter 23 prime canonical $\mathscr{L}_4$-terms are structured unary C-schemes, while in general $\mathscr{L}_4$-terms are unary recursive C-schemes. Therefore, every unary recursive C-scheme is equivalent by 26.11 to a structured unary C-scheme, provided its free variables are interpreted by predicates and functions which do not use or affect the second counter. The proofs of 9.15*, 21.10 actually give an algorithm which transforms recursive program schemes into recursion-free ones. While this result has already been mentioned in chapter 23, we point out below that it also holds for other classes of program schemes.

So far $\mathscr{L}_3$, $\mathscr{L}_4$-terms have been regarded as single-valued unary recursive schemes, which assumes that terms are interpreted by functions $\varphi: \omega \times M \longrightarrow \omega \times M$, respectively $\varphi: \omega^2 \times M \longrightarrow \omega^2 \times M$, interpreting predicate symbols by predicates $P: \omega \times M \longrightarrow \{0, 1\}$, respectively $P: \omega^2 \times M \longrightarrow \{0, 1\}$. However, examples 22.1, 21.1, 25.1, 25.2, 25.7 provide other interesting interpretations for $\mathscr{s}_1$ which can be extended to interpretations for $\mathscr{s}_3$, $\mathscr{s}_4$ by introducing appropriate kinds of predicates. (The third and fourth axioms for conditionals will have to be modified or dropped in some cases. These axioms as well as the first and fifth axioms for conditionals and $\theta O = O$ were used in no proof in this chapter.) Consequently, $\mathscr{L}_3$, $\mathscr{L}_4$-terms could also be regarded as multiple-valued, fuzzy, probabilistic etc. unary recursive schemes. Further details are left for the exercises.

Unary recursive schemes can be turned into n-ary ones by adding to the languages $\mathscr{L}_3$, $\mathscr{L}_4$ constants $I_i^{(j)}$, $1 \le i \ne j \le n$, as in chapter 23. Similarly, a stack can be introduced by adding to $\mathscr{L}_3$, $\mathscr{L}_4$ new constants Si_i, So_i, $1 \le i \le n$. While we are not able to suggest satisfactory axioms for $I_i^{(j)}$ or Si_i, So_i, no additional axioms are needed to reaffirm 26.6–26.11 except that the new constants should commute with L_2, R_2.

It is noteworthy that the interpretations discussed above and, perhaps, all those interesting from the standpoint of Computer Science validate the infinitary (£££)-rule

$$\frac{\alpha_n \varphi = \alpha_{n+1} R, n \ge 0}{\alpha_0 R[\varphi] = O}.$$

where the terms α_n are constructed from I, L, R by multiplication. Therefore, the structures used for those interpretations have the representation property so that all recursively axiomatizable extensions of $\mathscr{A}_0$ (including $\mathscr{A}_1, \mathscr{A}_3, \mathscr{A}_4$) are incomplete with respect to them by 26.2.

Writing a, b, c for formulas of a certain language and introducing the formula construct $\{a\}\varphi\{b\}$, one may use the following Hoare-style inference rules for proving partial correctness of structured $\mathscr{L}_1$-programs.

$$\frac{\{a\}\varphi\{c\}, \{c\}\psi\{b\}}{\{a\}\varphi\psi\{b\}},$$

$$\frac{\{a\}L_1\varphi\{b\}, \{a\}R_1\psi\{b\}}{\{a\}(\varphi, \psi)\{b\}},$$

$$\frac{\{a\}R_1\{a\}, \{a\}L_1\varphi L\{b\}, \{b\}R\{b\}}{\{a\}\langle\varphi\rangle\{b\}},$$

$$\frac{\{a\}L_1\{b\}, \{a\}R_1\varphi\{a\}}{\{a\}[\varphi]\{b\}},$$

where $L_1 = (I, O)$, $R_1 = (O, I)$. (Compare with Hoare [1969], Apt [1981].) If $\mathscr{L}_1$-terms are interpreted by single-valued functions, then $\{a\}\varphi\{b\}$ says that whenever $a(s)$ is true and $\varphi(s) = t$, then $b(t)$ is true. Inversely interpreted rules are helpful too, with $\{a\}\varphi\{b\}$ meaning that whenever $\varphi(s) = t$ and $b(t)$, then $a(s)$. The correctness of these two interpretations is shown in the exercises.

More generally, the systems considered in this chapter can also be related to certain varieties of Dynamic, Algorithmic, Process etc. Logics, thereby bringing these logics closer to 'real life'.

EXERCISES TO CHAPTER 26

Exercise 26.1. Given a system of inequalities $\varphi_i \leq \theta_i$, $1 \leq i \leq m$, where φ_i are (regular in $\theta_1, \ldots, \theta_m$ and regular) $\mathscr{L}_2$-terms, show that there are (regular) $\mathscr{L}_2$-terms $\sigma_1, \ldots, \sigma_m$ such that $\theta_1, \ldots, \theta_m$ are not free in them, $\vdash_{\mathscr{A}_2} \varphi_i(\sigma_1, \ldots, \sigma_m/\theta_1, \ldots, \theta_m) \leq \sigma_i$, $1 \leq i \leq m$, and $\vdash_{\mathscr{A}_2} \varphi_i \leq \theta_i$, $1 \leq i \leq m \rightarrow \sigma_i \leq \theta_i$, $1 \leq i \leq m$.

Hint. Induction on m. If $m = 1$, then take $\sigma_1 = \mu\theta_1.\varphi_1$. Given a system $\varphi_i \leq \theta_i$, $1 \leq i \leq m+1$, take $\sigma = \mu\theta_{m+1}.\varphi_{m+1}$, then take $\sigma_1, \ldots, \sigma_m$ to correspond to the system $\varphi_i(\sigma/\theta_{m+1}) \leq \theta_i$, $1 \leq i \leq m$ by the induction clause, and $\sigma_{m+1} = \sigma(\sigma_1, \ldots, \sigma_m/\theta_1, \ldots, \theta_m)$. (Respectively, use the fact that whenever φ, ψ are regular in $\theta_1, \ldots, \theta_m$ and regular, and φ is regular in θ_{m+1}, then $\varphi(\psi/\theta_{m+1})$ is regular in $\theta_1, \ldots, \theta_m$ and regular.)

Exercise 26.2. Show that for every (regular) $\mathscr{L}_2$-term σ there is a system $\varphi_i \leq \theta_i$, $1 \leq i \leq m$ such that φ_i are of the form $I, O, \theta_k\theta_l$ (respectively, $k \neq 1, \ldots, m$) or $(P \rightarrow \theta_k, \theta_l)$ and σ is the first component of the least solution of that system in the sense of exercise 26.1.

Hint. Use induction on the construction of σ.

Exercise 26.3. Given an interpretation of $\mathscr{L}_3$ such that the axioms and rules of $\mathscr{I}_3$ apart from Scott's rule are valid and condition (***) is satisfied, show that Scott's rule is also valid.

Hint. Follow the proof of 18.14.

Exercise 26.4. Let $\mathscr{S}$ be a (***)-complete OS such that $(L, R) \leq I$ and $\varphi O = O$ for all φ. Extend it to an interpretation of $\mathscr{I}_3$.

Hint. Take $(P \to \varphi, \psi) = (L\varphi, R\psi)$.

Remark. This construction is not as unsatisfactory as it may seem. Indeed, the equality $(P \to \varphi, \psi) = \tilde{P}(\varphi, \psi)$ suggests that the expressive power of $\mathscr{I}_3$ is hardly increased by the presence of predicate variables.

Exercise 26.5. Add a new function constant F to $\mathscr{L}_3$ and find an interpretation for which the axioms of $\mathscr{I}_3$ and (μ) are valid, while Scott's rule is not.

Hint. Take the IOS $\mathscr{S}'$ of exercise 19.5 and interpret terms by members of the subspace consisting of all φ' such that $\varphi'(O) = O$. Define $(P \to \varphi', \psi')$ as $(\bar{L}\varphi', \bar{R}\psi')$, where $\bar{L} = \lambda\varphi.\lambda s.\varphi(3s)$, $\bar{R} = \lambda\varphi.\lambda s.\varphi(3s+1)$. Take $L\theta \leq I$ for Ψ and interpret F as the element ρ' of the hint to exercise 19.5. Then $\Psi(O/\theta)$ and $\Psi \to \Psi(F(I, \theta)/\theta)$ are valid but $\Psi(\mu\theta.F(I, \theta)/\theta)$ fails.

Exercise 26.6. Let $\mathscr{S}$ be the IOS of example 21.1. Consider fuzzy predicates $P: M \times \{0, 1\} \to E$. Interpreting conditionals by $(P \to \varphi, \psi)(s, t) = \sup\{\inf\{P(s, 0), \varphi(s, t)\}, \inf\{P(s, 1), \psi(s, t)\}\}$ show that the axioms and rules of $\mathscr{I}_3$ are valid, provided $\geq$ is substituted for $=$ in the third and fourth axioms for conditionals.

Taking $E = \{\perp, \top\}$ in particular, one gets interpretations by multiple-valued functions and predicates.

Exercise 26.7. Let $\mathscr{I}$ be the IOS of example 25.1 (example 25.2). Consider probabilistic predicates $P: M \times \{0, 1\} \to [0, \infty]$ (respectively, $P: M \times \{0, 1\} \to [0, 1]$ such that $\forall s(P(s, 0) + P(s, 1) \leq 1)$). Interpreting conditionals by $(P \to \varphi, \psi)(s, t) = P(s, 0)\varphi(s, t) + P(s, 1)\psi(s, t)$, show that the axioms and rules of $\mathscr{I}_3$ apart from the third and fourth axioms for conditionals are valid.

Exercise 26.8. Prove that the Hoare rules suggested at the end of this chapter are valid, provided $\mathscr{L}_1$-terms are interpreted by members of example 22.2.

Hint. The rules for $\circ, \Pi$ are verified directly. Supposing $\{a\}R_1\{a\}$, $\{a\}L_1\varphi L\{b\}$ and $\{b\}R\{b\}$ true, show that the set $\mathscr{E} = \{\theta/\{a\}\theta\{b\}\}$ is closed under the mapping $\lambda\theta.(\varphi L, \theta R)$. Use 18.2 to get $\langle\varphi\rangle \in \mathscr{E}$. Similarly for [ ].

Exercise 26.9. Prove that the Hoare rules are valid, provided $\mathscr{L}_1$-terms are interpreted by members of example 22.1, where $\{a\}\varphi\{b\}$ is true iff whenever $t \in \varphi(s)$ and $a(s)$ is true, then so is $b(t)$ (or whenever $t \in \varphi(s)$ and $b(t)$, then $a(s)$).

Systems $\mathscr{I}_1^{st}, \mathscr{I}_3^{st}$. The systems $\mathscr{I}_1, \mathscr{I}_3$ extended by adding a new crop of function variables $x, x_1, x_2, \ldots,$ function constants $K_0 - K_4$, a term construct $St(\varphi)$, axioms $St(I)K_0 = K_0$, $\chi K_0 = (\chi L, \chi R)$, $St(St(I))K_1 = K_1$, $\chi_1\chi_2K_1K_2 = \chi_1\chi_2$, $K_3K_4 = I$, $\chi St(\theta) = \theta\chi$ where χ, χ_1, χ_2 range over $\mathscr{L}$-terms ($K_3, x, x_1, \ldots$ are $\mathscr{L}$-terms and whenever χ_1, χ_2 are $\mathscr{L}$-terms, then so is $\chi_1\chi_2K_1$), and the rule

$$\frac{\vec{\phi} \to \varphi x \psi \leq \rho x \sigma}{\vec{\phi} \to St(\varphi)\psi \leq St(\rho)\sigma},$$

where x does not occur in $\vec{\phi}, \varphi, \psi, \rho, \sigma$. Finally Ψ is allowed to stand for $St(\theta) \leq St(I)\tau$ in (μ).

Exercise 26.10. Prove that 26.6, 26.8 hold for $\mathscr{A}_3^{st}$. Establish an analogue to 26.7.

Hint. Show that St is a t-operation in $\mathscr{A}_1^{st}$.

CHAPTER 27

Skordev combinatory spaces

The present approach belongs to the algebraic-axiomatic trend in Recursion Theory started by Skordev [1976]. In a series of papers Skordev develops a general theory of recursion on the specially designed algebraic system of combinatory space and studies a wide variety of interesting spaces to represent certain particular concepts of effective computability, as well as to introduce such concepts in new areas. Most of the work on the subject has been carried out by Skordev himself, and his book Skordev [1980] offers a comprehensive presentation.

This chapter examines Skordev combinatory spaces from the point of view of operative spaces. Roughly speaking, the former are OS with a storing operation, with the translation operation eliminated in the case of iterative spaces.

We present the notion of combinatory space in a recent version proposed by Skordev [1980a] with somewhat different notation. A *combinatory space* is a 9-tuple

$$\mathscr{S}^* = (\mathscr{F}, I, \mathscr{C}, \Pi^*, L^*, R^*, \Sigma, T, F),$$

where $\mathscr{F}$ is a partially ordered semigroup with unit I, $\Pi^*: \mathscr{F}^2 \to \mathscr{F}$, $\Sigma: \mathscr{F}^3 \to \mathscr{F}$ is monotonic, $\mathscr{C} \subseteq \mathscr{F}$, $L^*, R^*, T, F \in \mathscr{F}$, $T \neq F$, $\mathscr{C}T \subseteq \mathscr{C}$, $\mathscr{C}F \subseteq \mathscr{C}$, $\Pi^*(\mathscr{C}, \mathscr{C}) \subseteq \mathscr{C}$ and

$$\begin{aligned}
\forall x(x\varphi \leq x\psi) &\Rightarrow \varphi \leq \psi, \\
(x, y)^*L^* = x,\ &(x, y)^*R^* = y, \\
x(\varphi, \psi)^* &= (x\varphi, x\psi)^*, \\
\varphi(I, x\psi)^* = (\varphi, x\psi)^*, &\quad \psi(x, I)^* = (x, \psi)^*, \\
x(\chi \to \varphi, \psi) &= (x\chi \to x\varphi, x\psi), \\
\chi(I \to x\varphi, x\psi) &= (\chi \to x\varphi, x\psi), \\
(\chi \to \varphi, \psi)\rho &= (\chi \to \varphi\rho, \psi\rho), \\
(T \to \varphi, \psi) = \varphi, &\quad (F \to \varphi, \psi) = \psi,
\end{aligned}$$

where $(\varphi, \psi)^*$, $(\chi \to \varphi, \psi)$ stand for $\Pi^*(\varphi, \psi)$, $\Sigma(\chi, \varphi, \psi)$ and x, y range over $\mathscr{C}$.

Take for example $\mathscr{F} = \{\varphi/\varphi: \omega - \to \omega\}$, $\mathscr{C} = \{\lambda t.s/s \in \omega\}$, the operations $\circ, \Pi^*$ and branching of chapter 2 for $\circ, \Pi^*, \Sigma$ and the elements $\psi_0 - \psi_4$ considered there for L^*, R^*, I, T, F. Then $\mathscr{S}^* = (\mathscr{F}, I, \mathscr{C}, \Pi^*, L^*,$

$R^*, \Sigma, T, F)$ is a combinatory space. Some higher order examples will be given in the exercises.

An element φ is *normal* iff $\mathscr{C}\varphi \subseteq \mathscr{C}$. For example, I, T, F are normal. It follows easily that all normal elements φ are left distributive with respect to Π^*, Σ, i.e. $\varphi(\rho, \sigma)^* = (\varphi\rho, \varphi\sigma)^*$, $\varphi(\chi \to \rho, \sigma) = (\varphi\chi \to \varphi\rho, \varphi\sigma)$ for all χ, ρ, σ.

The following assertions are due to Skordev.

Proposition 27.1. If φ, ψ are normal, then $(\varphi, \psi)^* L^* = \varphi$. If φ is normal, then $(\varphi, \psi)^* R^* = \psi$.

Proof. If φ, ψ are normal, then

$$x(\varphi, \psi)^* L^* = (x\varphi, x\psi)^* L^* = x\varphi$$

for all x; hence $(\varphi, \psi)^* L^* = \varphi$.

It follows that $y(x, I)^* R^* = (x, y)^* R^* = y$ for all y, hence $(x, I)^* R^* = I$. Therefore, if ϕ is normal, then

$$x(\varphi, \psi)^* R^* = (x\varphi, x\psi)^* R^* = x\psi(x\varphi, I)^* R^* = x\psi I = x\psi$$

for all x; hence $(\varphi, \psi)^* R^* = \psi$.

Proposition 27.2. $xy = y$.

Proof. $xy = x(I, y)^* R^* = (x, y)^* R^* = y$.

In particular, 27.2 implies that all the elements in $\mathscr{C}$ are normal.

Proposition 27.3. $(\rho, I)^*(L^*\chi \to R^*\varphi, R^*\psi) = (\rho\chi \to \varphi, \psi)$.

Proof. We have

$$\begin{aligned} x(\rho, I)^*(L^*\chi \to R^*\varphi, R^*\psi) &= x\rho(I, x)^*(L^*\chi \to R^*\varphi, R^*\psi) = x\rho(\chi \to x\varphi, x\psi) \\ &= (x\rho\chi \to x\varphi, x\psi) = x(\rho\chi \to \varphi, \psi) \end{aligned}$$

for all x, which completes the proof.

Proposition 27.4. $(xT \to \varphi, \psi) = \varphi$, $(xF \to \varphi, \psi) = \psi$.

Proof. Using 27.2, we get

$$y(xT \to \varphi, \psi) = (xT \to y\varphi, y\psi) = x(T \to y\varphi, y\psi) = y\varphi$$

for all y, hence $(xT \to \varphi, \psi) = \varphi$ and similarly $(xF \to \varphi, \psi) = \psi$. The proof is complete.

The components of $\mathscr{S}^*$ enable us to construct a companion OS $\mathscr{S}$.

Proposition 27.5. Take $\Pi(\varphi, \psi) = (L^* \to R^*\varphi, R^*\psi)$, $L = (T, I)^*$ and $R = (F, I)^*$. Then $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is an OS.

Proof. Π is obviously monotonic, while

$$\begin{aligned} (\varphi, \psi)\chi &= (L^* \to R^*\varphi, R^*\psi)\chi = (L^* \to R^*\varphi\chi, R^*\psi\chi) = (\varphi\chi, \psi\chi), \\ L(\varphi, \psi) &= (T, I)^*(L^* \to R^*\varphi, R^*\psi) = (T \to \varphi, \psi) = \varphi \end{aligned}$$

by 27.3 and similarly $R(\varphi, \psi) = \psi$, which completes the proof.

Proposition 27.6. $(\chi \to \varphi, \psi) = (\chi, I)^*(\varphi, \psi)$.

This follows from 27.3.

If the spaces $\mathscr{S}^*, \mathscr{S}$ are to be in some sense equivalent, then their initial operations should be expressible in terms of each other. Since there is nothing in $\mathscr{S}$ to match Π^*, we introduce the additional operation $St = \lambda\varphi.(L^*, R^*\varphi)^*$. Some elementary properties follow to show that St is a storing operation with a corresponding set $\mathscr{L} = \mathscr{C}^{\sim}$, where $\tilde{x} = (x, I)^*$. Notice that all $\tilde{x}$ are normal.

Proposition 27.7. $\tilde{x}St(\varphi) = \varphi\tilde{x}$.

Proof. $\tilde{x}(L^*, R^*\varphi)^* = (\tilde{x}L^*, \tilde{x}R^*\varphi)^* = (x, \varphi)^* = \varphi\tilde{x}$.

Proposition 27.8. If $\varphi\tilde{x}\psi \leq \rho\tilde{x}\sigma$ for all $\tilde{x}$, then $St(\varphi)\psi \leq St(\rho)\sigma$.

Proof. We have $(x, \varphi)^*\psi \leq (x, \rho)^*\sigma$ for all x. Multiplying on the left by yR^*, we get $(x, yR^*\varphi)^*\psi \leq (x, yR^*\rho)^*\sigma$ for all x, y; hence $(I, yR^*\varphi)^*\psi \leq (I, yR^*\rho)^*\sigma$. Multiplying on the left by yL^*, we get $(yL^*, yR^*\varphi)^*\psi \leq (yL^*, yR^*\rho)^*\sigma$ for all y, which implies $St(\varphi)\psi \leq St(\rho)\sigma$.

Proposition 27.9. $(\varphi, \psi)^* = K_0^*St(\varphi)K_1^*St(\psi)$, where $K_0^* = (I, I)^*$, $K_1^* = (R^*, L^*)^*$.

Proof. We have

$$\begin{aligned} xK_0^*St(\varphi)K_1^*St(\psi) &= x\tilde{x}St(\varphi)K_1^*St(\psi) = x\varphi\tilde{x}K_1^*St(\psi) = x\varphi(I, x)^*St(\psi) \\ &= x\varphi(I, x\psi)^* = (x\varphi, x\psi)^* = x(\varphi, \psi)^* \end{aligned}$$

for all x. The proof is complete.

Proposition 27.10. $\tilde{x}K_0 = (\tilde{x}L, \tilde{x}R)$, where $K_0 = St(I)(R^*L^* \to St(R^*)L, St(R^*)R)$.

Proof.

$$\begin{aligned} \tilde{x}K_0 &= (\tilde{x}R^*L^* \to \tilde{x}St(R^*)L,\ \tilde{x}St(R^*)R) \\ &= (L^* \to R^*\tilde{x}L,\ R^*\tilde{x}R) = (\tilde{x}L, \tilde{x}R). \end{aligned}$$

Proposition 27.11. $\tilde{x}\tilde{y}K_1 = (y, x)^{*\sim}$, where $K_1 = St^2(I)St(K_1^*)(R^*L^*, St(R^*))^*K_1^*$.

Proof.

$$\tilde{x}\tilde{y}K_1 = (y, I, x)^*(R^*L^*, St(R^*))^*K_1^* = (I, y, x)^*K_1^* = ((y, x)^*, I)^* = (y, x)^{*\sim}.$$

Proposition 27.12. $(y, x)^{*\sim}K_2 = \tilde{x}\tilde{y}$, where $K_2 = K_1^*(R^*L^*, St(R^*))^*St(K_1^*)$.

Proof.

$$\begin{aligned} (y, x)^{*\sim}K_2 &= (I, y, x)^*(R^*L^*, St(R^*))^*St(K_1^*) = (y, I, x)^*St(K_1^*) \\ &= (y, x, I)^* = \tilde{x}\tilde{y}. \end{aligned}$$

Proposition 27.13. St is a storing operation.

This follows from 27.7, 27.8, 27.10–27.12.

An element φ is *polynomial* in* $\mathscr{B}\subseteq\mathscr{F}$ iff $\varphi\in cl(\{I,L^*,R^*,T,F\}\cup\mathscr{B}/\circ,\Pi^*,\Sigma)$.

Proposition 27.14. The following are equivalent.

(1) φ is polynomial* in $\mathscr{B}$.

(2) φ is polynomial in $\mathscr{B}_0\cup St(\mathscr{B})$, where

$$\mathscr{B}_0=\{R^*, St(T), St(F), St(K_0^*), St(K_1^*), St^2(L^*), St^2(R^*)\}.$$

Proof. The implication (2)⇒(1) is immediate. Assuming (1), an easy induction on the construction of φ using 27.9, 27.6, 10.12, 10.13, 10.16 shows that $St(\varphi)$ is polynomial in $\mathscr{B}_0\cup St(\mathscr{B})$; hence so is φ since $\varphi=LSt(\varphi)R^*$. The proof is complete.

A subset $\mathscr{E}$ of $\mathscr{F}$ is a *regular* segment* iff $\mathscr{E}=\{\theta/\chi\theta\rho\leq\tau$ for all $\langle\chi,\rho,\tau\rangle\in\mathscr{A}\}$ for a certain $\mathscr{A}\subseteq\{\varphi/\mathscr{C}\varphi\subseteq\mathscr{C}\}\times\mathscr{F}^2$. Assume from now on that the space $\mathscr{S}^*$ is *iterative* in the sense of Skordev [1980]. That is, an *iteration operation* $[\ ,\ ]:\mathscr{F}^2\to\mathscr{F}$ exists for which $(\psi\to I,\varphi[\varphi,\psi])=[\varphi,\psi]$ and $[\varphi,\psi]$ is a member of all regular* segments closed under $\lambda\theta.(\psi\to I,\varphi\theta)$.

Proposition 27.15. The companion OS $\mathscr{S}$ of $\mathscr{S}^*$ is iterative. Moreover, the axiom μA_3 holds for mappings $\lambda\theta.(I,\varphi\theta)$ and the assertion of 5.7** holds.

Proof. Taking $[\varphi]=[R^*\varphi,L^*]R^*$, we have

$$\begin{aligned}(I,\varphi[\varphi])&=(L^*\to R^*,R^*\varphi[R^*\varphi,L^*]R^*)=(L^*\to I,R^*\varphi[R^*\varphi,L^*])R^*\\&=[R^*\varphi,L^*]R^*=[\varphi].\end{aligned}$$

Let us show that $[\varphi]$ is a member of any regular segment $\mathscr{E}$ closed under $\lambda\theta.(I,\varphi\theta)$. Consider the regular* segment $\mathscr{E}^*=\{\theta/\theta R^*\in\mathscr{E}\}$. If $\theta\in\mathscr{E}^*$, then $\theta R^*\in\mathscr{E}$; hence $(L^*\to I,R^*\varphi\theta)R^*=(L^*\to R^*,R^*\varphi\theta R^*)=(I,\varphi\theta R^*)\in\mathscr{E}$, which implies $(L^*\to I,R^*\varphi\theta)\in\mathscr{E}^*$. Therefore, $[R^*\varphi,L^*]\in\mathscr{E}^*$; hence $[\varphi]\in\mathscr{E}$.

In order to apply 5.13, we construct an operation $\langle\quad\rangle_1$ such that $\bar{n}\langle\varphi\rangle_1=\varphi\bar{n}$ for all n,φ.

Let $\sigma=(L^{*2}\to(L^*R^*,R^*R)^*L,\ (L^*R^*,R^*R)^*R)$. Then

$$\begin{aligned}x(\bar{0},\varphi\bar{k})^*\sigma&=(x\bar{0},x\varphi\bar{k})^*\sigma=x\varphi\bar{k}(x\bar{0},I)^*\sigma=x\varphi\bar{k}(xT\to(x,R)^*L,(x,R)^*R)\\&=x\varphi\bar{k}(x,R)^*L=(x,x\varphi\overline{k+1})^*L=x(I,\varphi\overline{k+1})^*L\end{aligned}$$

for all x; hence $(\bar{0},\varphi\bar{k})^*\sigma=(I,\varphi\overline{k+1})^*L$, while

$$\begin{aligned}x(\overline{n+1},\varphi\bar{k})^*\sigma&=x\varphi\bar{k}(x\overline{n+1},I)^*\sigma=x\varphi\bar{k}(x\bar{n}F\to(x\bar{n},R)^*L,(x\bar{n},R)^*R)\\&=x\varphi\bar{k}(x\bar{n},R)^*R=(x\bar{n},x\varphi\overline{k+1})^*R\end{aligned}$$

for all x; hence $(\overline{n+1},\varphi\bar{k})^*\sigma=(\bar{n},\varphi\overline{k+1})^*R$.

Take $\langle\varphi\rangle_1=(L^*,R^*,[I]\varphi\bar{0})^*[\sigma]R^*$. Using the easy equalities $(T,\rho)^*\circ[\sigma]=\rho$, $(\bar{n}F,\rho)^*[\sigma]=\rho\sigma[\sigma]$, we get

$$\begin{aligned}\bar{0}\langle\varphi\rangle_1&=(T,I,\varphi\bar{0})^*[\sigma]R^*=(I,\varphi\bar{0})^*R^*=\varphi\bar{0},\\ \overline{n+1}\langle\varphi\rangle_1&=(\bar{n}F,\bar{n},\varphi\bar{0})^*[\sigma]R^*\\&=(\bar{n},\varphi\bar{0})^*\sigma[\sigma]R^*=(\overline{n-1},\varphi\bar{1})^*\sigma[\sigma]R^*=\cdots=(\bar{0},\varphi\bar{n})^*\sigma[\sigma]R^*\end{aligned}$$

$$=(I,\varphi\overline{n+1})^*L[\sigma]R^*=(I,\varphi\overline{n+1})^*R^*=\varphi\overline{n+1};$$

hence $\bar{n}\langle\varphi\rangle_1=\varphi\bar{n}$ for all n. By 5.13, the proof is complete.

Proposition 27.16. $[\varphi,\psi]=(\psi,I)^*[\varphi(\psi,I)^*]$.

Proof. Using 27.6 and 6.10, we get

$$[\varphi,\psi]=\mu\theta.(\psi\to I,\varphi\theta)=\mu\theta.(\psi,I)^*(I,\varphi\theta)=(\psi,I)^*[\varphi(\psi,I)^*].$$

Proposition 27.17. *St* is a t-operation whose corresponding set of functional elements is the set $\mathscr{B}_0$ of 27.14. The additional assumptions of exercises 10.2–10.5 are also satisfied.

Proof. The mappings $\Sigma_0=\lambda\theta_1\theta_2.\theta_1\theta_2$, $\Sigma_1=\lambda\theta_1\theta_2.K_0(\theta_1,\theta_2)$, $\Sigma_3=\lambda\theta.K_0[\theta K_0]St(I)$, $\Sigma_4=\lambda\theta.K_1\theta K_2$ and $\Sigma_5=\lambda\theta.L\theta R^*$ are price recursive in $\mathscr{B}_0$ and satisfy the equalities (0), (1), (3)–(5) of chapter 10 by 27.15, 10.18. By 27.14, the proof of 27.15 gives a mapping Σ_7 prime recursive in $\mathscr{B}_0$ such that $c(\Sigma_7)=1$ and $\langle\varphi\rangle=\Sigma_7(St(\varphi))$ for all φ. The proof is complete.

Now we are able to show that recursiveness of $\mathscr{S}^*$ and st-recursiveness of $\mathscr{S}$ are equivalent. The former will be marked by an asterisk to distinguish it from IOS-recursiveness of $\mathscr{S}$. Of course, an element φ is *recursive* in* $\mathscr{B}$ iff

$$\varphi\in cl(\{I,L^*,R^*,T,F\}\cup\mathscr{B}/\circ,\Pi^*,\Sigma,[\ ,\ ]).$$

The notion of mapping recursive* in $\mathscr{B}$ is introduced via parametrization as usual.

Proposition 27.18. Let $\Gamma:\mathscr{F}^n\to\mathscr{F}$, $n\geq 1$, and $\mathscr{B}\subseteq\mathscr{F}$. Then the following are equivalent.

(1) Γ is recursive* in $\mathscr{B}$.

(2) Γ is st-recursive in $\mathscr{B}$.

(3) Γ is prime st-recursive in $\mathscr{B}$.

The same equivalences hold for elements.

Proof. Propositions 27.9, 27.6, 27.16 ensure (1)$\Rightarrow$(2), while 27.17 gives (2)$\Rightarrow$(3). The implication (3)$\Rightarrow$(1) is obvious. This completes the proof.

The last statement makes it possible to derive some basic theorems of the theory of Skordev combinatory spaces as particular instances of results established in chapter 10. These include a Normal Form Theorem, an Enumeration Theorem, a Secton Recursion Theorem etc. The space $\mathscr{S}$ is (£££)-iterative, which gives by 8.3*** a Representation Theorem for $\mathscr{S}^*$ and yields the implications of 26.1–26.3.

Assume that $\mathscr{S}^*$ satisfies a slightly stronger μ-axiom, namely that every unary mapping Γ recursive* in $\mathscr{F}$ has a fixed point which is a member of all regular* segments closed under Γ. Then a First Recursion Theorem for $\mathscr{S}^*$ is also furnished by 10.8* (or exercise 10.5) too.

It follows that in this case the companion space $\mathscr{S}$ satisfies the axiom $t\mu A_3$ and the stronger $t\mu$-axiom of exercise 10.9. The other assumptions of exercise 10.9 are also satisfied, provided one takes

$$K_4=R^*,\quad K_5=(L^*,L^*,R^*)^*,\quad K_6=(R^*L^*,L^*,R^{*2})^*.$$

While 27.5, 27.15 guarantee that all (iterative) combinatory spaces have companion (iterative) OS based on the same semigroups, the converse fails. Therefore, there are more OS than combinatory spaces.

Proposition 27.19. There is an IOS $\mathscr{S}$ whose semigroup $\mathscr{F}$ is a semigroup of no combinatory space.

Proof. Let $\mathscr{S}_0 = (\mathscr{F}_0, I, \Pi_0, L, R)$ be the IOS of example 4.3 and $\mathscr{F} = \{\varphi \in \mathscr{F}_0 / \varphi$ is recursive$\}$. Then $\mathscr{S} = (\mathscr{F}, I, \Pi_0 \restriction \mathscr{F}^2, L, R)$ is a μA_3-iterative subspace of $\mathscr{S}_0$ by exercise 18.6.

Suppose that $\mathscr{F}$ is a semigroup of a certain combinatory space $\mathscr{S}^* = (\mathscr{F}, I, \mathscr{C}, \Pi^*, L^*, R^*, \Sigma, T, F)$. Then $I \notin \mathscr{C}$ since $I \in \mathscr{C}$ would give $\mathscr{C} = \{I\}$ by 27.2, hence $T = F$, which is not the case. We shall use the unwinding method to show that whenever $\varphi \neq I$ and $\varphi\varphi = \varphi$, then $\varphi(L, R) = \varphi$. In particular, $x(L, R) = x$ for all $x \in \mathscr{C}$, hence $(L, R) = I$. That will complete the proof since the last equality is false.

Let φ be a recursive member of $\mathscr{F}_0$, $\varphi \neq I$ and $\varphi\varphi = \varphi$. Then $\varphi = \bar{1}[\sigma]$ with a certain primitive σ by 9.3. It follows easily that for all $\alpha \in \mathscr{D}$ either $\alpha\sigma \in \mathscr{D}$ or $\alpha\sigma \in \Pi(\mathscr{F}, \mathscr{F})$. Start constructing a sequence $\{\alpha_n\}$ such that $\alpha_0 = \bar{0}$, $\alpha_n\sigma = \alpha_{n+1}R$ and $\varphi = \alpha_n\sigma[\sigma]$. Then either $\alpha_n\sigma \in \mathscr{D}$ or $\alpha_n\sigma \in \Pi(\mathscr{F}, \mathscr{F})$ at the n-th step. If $\alpha_n\sigma = \beta R$ for a certain β, then take $\alpha_{n+1} = \beta$ and continue. Otherwise stop.

Suppose that $\alpha_n\sigma = \beta L$ for a certain β. Then $\varphi = \beta L[\sigma] = \beta$; hence $\varphi(L, R) = \varphi$ by $\varphi \neq I$.

Suppose that $\alpha_n\sigma = I$ or $\alpha_n\sigma \in \Pi(\mathscr{F}, \mathscr{F})$. Then $\varphi \in \Pi(\mathscr{F}, \mathscr{F})$; hence $\varphi(3k+2)\uparrow$ for all k. If $\varphi(s)$ is not of the form $3k+2$, then $\varphi(L, R)(s) = \varphi(s)$. Supposing $\varphi(s) = 3k+2$, one gets $\varphi(\varphi(s))\uparrow$ contrary to the equality $\varphi(\varphi(s)) = \varphi(s)$. Therefore, $\varphi(L, R)(s) = \varphi(s)$ for all s; hence $\varphi(L, R) = \varphi$.

If an infinite sequence $\{\alpha_n\}$ is obtained, then $\varphi = O$ by (£££); hence $\varphi(L, R) = \varphi$ again. This completes the proof.

To further elucidate the connections between OS and Skordev combinatory spaces it is desirable to characterize those OS augmented by the storing operation which have companion combinatory spaces.

EXERCISES TO CHAPTER 27

The first five exercises are due to D. Skordev.

Exercise 27.1. Let $\mathscr{S}^*$ be a combinatory space. Prove the following equalities.

a. $((\chi \to \varphi, \psi), \rho)^* = (\chi \to (\varphi, \rho)^*, (\psi, \rho)^*)$.

b. $((\chi \to \varphi, \psi) \to \rho, \sigma) = (\chi \to (\varphi \to \rho, \sigma), (\psi \to \rho, \sigma))$.

c. $(\rho \to (\sigma \to \varphi, \psi), (\sigma \to \chi, \tau)) = (\sigma \to (\rho \to \varphi, \chi), (\rho \to \psi, \tau))$.

d. $(I, \varphi)^* K_1^* = (\varphi, I)^*, (\varphi, I)^* K_1^* = (I, \varphi)^*$.

e. $(\varphi, \psi)^* St(\chi) = (\varphi, \psi\chi)^*$.

Exercise 27.2. Assuming that $\mathscr{S}^*$ meets the stronger axiom $\psi(x\varphi, I)^* = (x\varphi, \psi)^*$, prove the following equalities.

a. $\rho(x\chi \to \varphi, \psi) = (x\chi \to \rho\varphi, \rho\psi)$.
b. $(\varphi, \psi)^* K_1^* = (\psi, \varphi)^*$.
c. $(\varphi, \psi)^*(L^*\rho, R^*\sigma)^* = (\varphi\rho, \psi\sigma)^*$.
Hint for a. Use exercise 27.1a to get

$$\rho(x\chi \to \varphi, \psi) = \rho(x(\chi \to T, F), I)^*(\varphi, \psi) = (x(\chi \to T, F), \rho)^*(\varphi, \psi)$$
$$= (x\chi \to \rho\varphi, \rho\psi).$$

Exercise 27.3. Let $\mathscr{S}^*$ be iterative and let $\mathscr{S}$ be its companion IOS. Prove that $(\chi \to \varphi, \psi) = (\chi, R)^*[R^*\psi L][\varphi L]$.

Remark. It follows in particular that $(\varphi, \psi) = (L^*, R)^*[R^{*2}\psi L][R^*\varphi L]$; hence the operations Σ, Π can be eliminated respectively in $\mathscr{S}^*, \mathscr{S}$.

In the following two exercises $\mathscr{S}^*$ is assumed to satisfy the axiom of exercise 27.2 and to have an element U^* such that $x, L^*, R^* \le U^*$, $U^*x \le x$ for all x. Such an element exists in the spaces corresponding to examples 21.1–21.3 but not examples 21.4, 22.2, 22.4, 25.1–25.4. In the sequel $\tilde{U}$ stands for $(U^*, I)^*$.

Exercise 27.4. Prove the following assertions.
a. $U^*x = x$, $\tilde{U}R^* = I$.
b. $\forall x(\tilde{x}\varphi \le \tau) \Rightarrow \tilde{U}\varphi \le \tau$.
c. $xU^* = U^*$.
d. $\tilde{U}St(\varphi) = \varphi\tilde{U}$.
e. $\tilde{U}K_0 = (\tilde{U}L, \tilde{U}R)$.
f. $\tilde{U}\tilde{U} = \tilde{U}K_2$.
g. $St(\tilde{U}) = \tilde{U}St^2(I)(R^*L^*, St(R^*))^*$.
Hint for b. If $\forall x(\tilde{x}\varphi \le \tau)$, then $St(\varphi) \le St(I)R^*\tau$ by (\$). Multiply on the left by $\tilde{U}$.
Hint for c. $\forall x(\varphi \le x\psi)$ implies $\forall x(x\varphi \le \psi)$.

Exercise 27.5. Show that φ is recursive* in $\{U^*\} \cup \mathscr{B}$ iff there is a ψ recursive* in $\mathscr{B}$ such that $\varphi = \tilde{U}\psi$.

Hint. Using exercise 27.4, show that $\varphi = \tilde{U}R^*\varphi$, $\tilde{U}\varphi\tilde{U}\psi = \tilde{U}K_2St(\varphi)\psi$, $(\tilde{U}\varphi, \tilde{U}\psi) = \tilde{U}K_0(\varphi, \psi)$, $St(\tilde{U}\varphi) = \tilde{U}St^2(I)(R^*L^*, St(R^*))^*St(\varphi)$ and $[\tilde{U}\varphi] = \tilde{U}K_0[K_2St(\varphi)K_0]R^*$. For the last equality observe that $[\sigma]$ is a member of all regular* segments closed under $\lambda\theta.(I, \sigma\theta)$ by the proof of 27.15.

Remark. This exercise establishes a normal form for search computability by 24.4, 27.18.

Exercise 27.6. Let (M, J, L', R') be a pairing space, let E be a partially ordered set with at least two distinct members, let $\Omega: M \times E^2 \to E$ and let $T, F: M \to M$ be monotonic such that $\Omega(T(s), e, d) = e$, $\Omega(F(s), e, d) = d$ for all $s \in M, e, d \in E$. Take

$$M' = \{\varphi/\varphi: M \to E \,\&\, \varphi \text{ is monotonic}\},$$
$$\mathscr{F}_1 = \{\phi/\phi: M \times M' \to E \,\&\, \phi \text{ is monotonic}\},$$

where ϕ is monotonic iff whenever $s \leq t$, $\varphi \leq \psi$, then $\phi(s,\varphi) \leq \phi(t,\psi)$. Also take

$$\mathscr{C}_1 = \{c^* = \lambda s\varphi.\varphi(c)/c \in M\},\ \phi \leq \Psi \text{ iff } \forall s\varphi(\phi(s,\varphi) \leq \Psi(s,\varphi)),$$
$$\phi\Psi = \lambda s\varphi.\phi(s, \lambda t.\Psi(t,\varphi)),$$
$$\Pi^*(\phi,\Psi) = \lambda s\varphi.\phi(s,\lambda t.\Psi(s,\lambda r.\varphi(J(t,r)))),$$
$$\Sigma(X,\phi,\Psi) = \lambda s\varphi.X(s,\lambda t.\Omega(t,\phi(s,\varphi),\Psi(s,\varphi))),$$
$$I_1 = \lambda s\varphi.\varphi(s),\ L^* = \lambda s\varphi.\varphi(L'(s)),\ R^* = \lambda s\varphi.\varphi(R'(s)),$$

$T_1 = \lambda s\varphi.\varphi(T(s))$ and $F_1 = \lambda s\varphi.\varphi(F(s))$. Show that $\mathscr{S}^* = (\mathscr{F}_1, I_1, \mathscr{C}_1, \Pi^*, L^*, R^*, \Sigma, T_1, F_1)$ is a combinatory space.

Remark. Smaller M' and $\mathscr{F}_1$ may also be taken provided the above definitions still make sense. This generalizes a construction suggested in Skordev [1980a], examples 4, 5, which can be obtained by taking the pairing space of 16.1 for (M, J, L', R').

Exercise 27.7. Let $\mathscr{S}^*$, E be as above and suppose that all well ordered subsets of E have least upper bounds. Show that $\mathscr{S}^*$ is iterative.

Hint. Following the proof of 18.13, show that every monotonic mapping $\Gamma:\mathscr{F}_1 \to \mathscr{F}_1$ has a fixed point which is a member of all regular* segments closed under Γ.

Exercise 27.8. Let (M, J, L', R'), E, Ω, T, F be the same as in exercise 27.6 and N be a nonempty set. Take

$$M' = \{\varphi/\varphi: M \times N \to E \,\&\, \varphi \text{ is monotonic}\},$$
$$\mathscr{F}_1 = \{\phi/\phi: M \times N \times M' \to E \,\&\, \phi \text{ is monotonic}\}.$$

Modify the construction of exercise 27.6 to get a combinatory space $\mathscr{S}^*$ based on $\mathscr{F}_1$ and prove that it is iterative, provided all the well ordered subsets of E have least upper bounds.

Hint. Take $c^* = \lambda sx\varphi.\varphi(c,x)$,

$$\phi\Psi = \lambda sx\varphi.\phi(s,x,\lambda ty.\Psi(t,y,\varphi)),$$
$$(\phi,\Psi)^* = \lambda sx\varphi.\phi(s,x,\lambda ty.\Psi(s,y,\lambda rz.\varphi(J(t,r),z))),$$
$$(X \to \phi,\Psi) = \lambda sx\varphi.X(s,x,\lambda ty.\Omega(t,\phi(s,x,\varphi),\Psi(s,x,\varphi))) \quad \text{etc.}$$

The following exercise shows that hierarchy OS are rich enough to have companion combinatory spaces.

Exercise 27.9. Let $\mathscr{S}, \mathscr{S}', \mathscr{S}''$ be consecutive OS and suppose that $\mathscr{S}''$ admits a transfer operation Tf. Using exercise 27.6, construct a combinatory space $\mathscr{S}^*$ based on $\mathscr{F}''$ such that whenever $\mathscr{S}^*$ is iterative, then so is $\mathscr{S}''$ and relative recursiveness* is equivalent to (prime) tf-recursiveness.

Hint. Take $E = M = \mathscr{F}$, $J = \Pi$, $\Omega(\chi,\varphi,\psi) = \chi(\varphi,\psi)$, $T = \check{L}$, $F = \check{R}$, $M' = \mathscr{F}'$ and $\mathscr{F}_1 = \{\phi:\mathscr{F} \times \mathscr{F}' \to \mathscr{F}/\lambda\theta'.\lambda\theta.\phi(\theta,\theta') \in \mathscr{F}''\}$; one need not distinguish between $\mathscr{F}_1$ and $\mathscr{F}''$. The operation Π^* is correctly introduced by exercise 14.2. Assuming $\mathscr{S}^*$ iterative, construct a companion IOS $\mathscr{S}_1$ by 27.5, 27.15. Use exercise 14.1 to show that $Tf = St$. Apply 27.13, then pass from $\mathscr{S}_1$ to $\mathscr{S}''$ by exercise 7.2.

CHAPTER 28

Recursive functionals

This chapter deals with some initial concepts and results of a recursion theory on monotonic functionals comparable with the theory developed in the first part of Kechris and Moschovakis [1977], thus paving the way for an adequate approach to Kleene-recursiveness in higher types presented in the next chapter. There is also a direct connection with the topics of chapter 30.

Given two arbitrary sets M, N, we recall that $\varphi: M \dashrightarrow N$ is a single-valued *function* and write $\mathscr{F}$ for the set of all such functions. A monotonic single-valued *functional* is a $\phi: M \times \mathscr{F} \dashrightarrow N$ such that whenever $\phi(s, \varphi) = u$ and $\varphi \leq \psi$, then $\phi(s, \psi) = u$. We are interested both in functions and functionals recursive in functionals, where the relevant notions of recursiveness are to be provided by an appropriate IOS.

Proposition 28.1 (Example 28.1). Let M, N be nonempty sets and f_1, f_2 be a splitting scheme for M. Take

$$\begin{aligned}
&\mathscr{F}_1 = \{\phi / \phi: M \times \mathscr{F} \dashrightarrow N \,\&\, \phi \text{ is monotonic}\},\\
&\phi \leq \Psi \text{ iff } \Psi \text{ is an extension of } \phi,\\
&\phi\Psi = \lambda s\varphi.\phi(s, \lambda t.\Psi(t, \varphi)),\\
&\Pi_1(\phi, \Psi)(f_1(s), \varphi) = \phi(s, \varphi),\\
&\Pi_1(\phi, \Psi)(f_2(s), \varphi) = \Psi(s, \varphi)
\end{aligned}$$

and $\Pi_1(\phi, \Psi)(s, \varphi)\uparrow$ otherwise, $I_1 = \lambda s\varphi.\varphi(s)$, $L_1 = \lambda s\varphi.\varphi(f_1(s))$ and $R_1 = \lambda s\varphi.\varphi(f_2(s))$. Then $\mathscr{S}_1 = (\mathscr{F}_1, I_1, \Pi_1, L_1, R_1)$ is a $(**)_0$-complete OS.

Proof. Taking $\Pi(\varphi, \psi)(f_1(s)) = \varphi(s)$, $\Pi(\varphi, \psi)(f_2(s)) = \psi(s)$ and $\Pi(\varphi, \psi)(s)\uparrow$ otherwise, $L' = \lambda\varphi.\lambda s.L_1(s, \varphi)$ and $R' = \lambda\varphi.\lambda s.R_1(s, \varphi)$, the quadruple $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ is a subspace of the pairing space of 16.4 with $E = \{\perp, \top\}$, $e = \perp$, restricting oneself to single-valued φ. Proposition 16.10 implies that $\mathscr{S}$ is strongly complete.

Let $\mathscr{S}'$ be obtained from $\mathscr{S}$ by 19.9. Then $\mathscr{S}'$ is a $(**)_0$-complete OS; hence so is its isomorphic copy $\mathscr{S}_1$, assigning to each mapping $\varphi': \mathscr{F} \to \mathscr{F}$ the functional $\lambda s\varphi.\varphi'(\varphi)(s)$. The proof is complete.

While the operations translation and iteration of $\mathscr{S}_1$ are introduced by $\langle\phi\rangle = \mu\Theta.(\phi L_1, \Theta R_1)$, $[\phi] = \mu\Theta.(I_1, \phi\Theta)$, the following statement characterizes them explicitly in the spirit of 3.4. Indeed, example 3.2 is a particular instance of example 28.1.

Proposition 28.2. Let $\mathscr{S}_1$ be the IOS of example 28.1. Then

$$\langle\phi\rangle(f_2^n(f_1(s)), \varphi) = \phi(s, \lambda t.\varphi(f_2^n(f_1(t)))),$$
$$\Delta(\phi, \Psi)(f_2^n(f_1(s)), \varphi) = \phi\Psi^n(s, \varphi)$$

and $\langle\phi\rangle(s, \varphi)$, $\Delta(\phi, \Psi)(s, \varphi)\uparrow$ otherwise. The function $\sigma = \lambda s.[\phi](s, \varphi)$ is the least satisfying the equalities

$$\sigma(f_1(s)) = \varphi(s), \quad \sigma(f_2(s)) = \phi(s, \sigma).$$

In other words, $\sigma = \mu\theta.(\varphi, \lambda s.\phi(s, \theta))$.

Proof. The operation Π_1 is continuous by exercise 18.1 since $(L_1, R_1) \leq I_1$. Therefore, $\Delta(\phi, \Psi) = \sup_m \Theta_m$ by exercise 18.4, where $\Theta_0 = O_1 = \lambda s\varphi.\uparrow$ and $\Theta_{m+1} = (\phi, \Theta_m\Psi)$. An easy induction on m gives that $\Theta_m(f_2^n(f_1(s)), \varphi) = \phi\Psi^n(s, \varphi)$, if $n < m$, and $\Theta_m(s, \varphi)\uparrow$ otherwise, which implies the desired characterization of Δ and that of $\langle\ \ \rangle$ by 6.32.

The characterization of [ ] follows from the proof of 12.25. This completes the proof.

Example 28.2. Let M, N_1, N be nonempty sets and let f_1, f_2 be a splitting scheme for M. Take the IOS $\mathscr{S}_1 = (\mathscr{F}_1, I_1, \Pi_1, L_1, R_1)$ of example 28.1 with $M \times N_1$, $\lambda sx.(f_1(s), x)$, $\lambda sx.(f_2(s), x)$ playing the role of M, f_1, f_2. Then

$$\mathscr{F} = \{\varphi/\varphi{:}M \times N_1 \dashrightarrow N\},$$
$$\mathscr{F}_1 = \{\phi/\phi{:}M \times N_1 \times \mathscr{F} \dashrightarrow N \,\&\, \phi \text{ is monotonic}\},$$
$$\phi\Psi = \lambda sx\varphi.\phi(s, x, \lambda ty.\Psi(t, y, \varphi)),$$
$$(\phi, \Psi)(f_1(s), x, \varphi) = \phi(s, x, \varphi),$$
$$(\phi, \Psi)(f_2(s), x, \varphi) = \Psi(s, x, \varphi),$$

and $(\phi, \Psi)(s, x, \varphi)\uparrow$ otherwise, $I_1 = \lambda sx\varphi.\varphi(s, x)$, $L_1 = \lambda sx\varphi.\varphi(f_1(s), x)$ and $R_1 = \lambda sx\varphi.\varphi(f_2(s), x)$.

Proposition 28.3. The initial operations of example 28.2 may be characterized as follows.

$$\langle\phi\rangle(f_2^n(f_1(s)), x, \varphi) = \phi(s, x, \lambda ty.\varphi(f_2^n(f_1(t)), y)),$$
$$\Delta(\phi, \Psi)(f_2^n(f_1(s)), x, \varphi) = \phi\Psi^n(s, x, \varphi)$$

and $\langle\phi\rangle(s, x, \varphi)$, $\Delta(\phi, \Psi)(s, x, \varphi)\uparrow$ otherwise,

$$\lambda sx.[\phi](s, x, \varphi) = \mu\theta.(\varphi, \lambda sx.\phi(s, x, \theta)).$$

This follows from 28.2.

While example 28.2 is a particular instance of example 28.1, an isomorphic copy of the latter can be obtained from the former by taking a singleton for N_1. In the sequel we shall speak about example 28.1 in some instances and example 28.2 in others, bearing this connection in mind.

A general theory of recursive functionals could have its starting point in the theory developed in part B of this book. The central notion is that of a functional ϕ recursive in a collection of functionals $\mathscr{B}_1 \subseteq \mathscr{F}_1$, for which the following three characteristic theorems hold in paticular.

Every functional ϕ when recursive in $\mathscr{B}_1$ has a normal form $\phi = \bar{1}[\Sigma]$ with a certain Σ primitive in $\mathscr{B}_1$ by 9.3. A similar normal form is assumed in the very definition of the corresponding notion of $\mathscr{I}$-recursiveness introduced by Kechris and Moschovakis [1977].

Whenever $\mathscr{B}_1$ is finite, then there is by the Enumeration Theorem 9.18 a functional Σ recursive in $\mathscr{B}_1$ such that every functional ϕ recursive in $\mathscr{B}_1$ equals $\bar{n}\Sigma$ for a certain n.

Whenever a mapping $\Gamma:\mathscr{F}_1 \to \mathscr{F}_1$ is recursive in $\mathscr{B}_1$, then its least fixed point $\mu\Theta.\Gamma(\Theta)$ is recursive in $\mathscr{B}_1$ by 9.13*. This assertion seems stronger than the First Recursion Theorem (also called Induction Completeness Theorem) 3.3 of the cited work, which is rather an analogue to 6.16. Take for example $\phi = \mu\Theta.(I_1, \Psi\Theta^2)$, the least functional satisfying the equalities $\Theta(f_1(s), \varphi) = \varphi(s), \Theta(f_2(s), \varphi) = \Psi(s, \lambda t.\Theta(t, \lambda r.\Theta(r, \varphi)))$. Proposition 9.13* implies that ϕ is recursive in Ψ, while the First Recursion Theorem of Moschovakis says nothing about ϕ.

Among the interesting functionals to be taken in $\mathscr{B}_1$ are those embodying quantification. Given a monotonic quantifier Q over M, we define a corresponding functional Q_M,

$$Q_M(s, \varphi) = \begin{cases} L_1 R_1(s, \varphi), & \text{if } Qt(L_1(t, \varphi) = 0), \\ R_1^2(s, \varphi), & \text{if } Q^{\cup}t \neg (L_1(t, \varphi) = 0), \\ \uparrow, & \text{otherwise,} \end{cases}$$

where 0 is a fixed number of N. Notice that

$$Q_M(X, \phi, \Psi)(s, \varphi) = \begin{cases} \phi(s, \varphi), & \text{if } Qt(X(t, \varphi) = 0), \\ \Psi(s, \varphi), & \text{if } Q^{\cup}t \neg (X(t, \varphi) = 0), \\ \uparrow, & \text{otherwise.} \end{cases}$$

In fact, Q_M is the mapping Q_σ of exercise 13.1 slightly modified to overcome the obstacle that $\mathscr{S}$ is a pairing space rather than an OS. Similarly, if $N_1 = N' \times N''$ and Q is a monotonic quantifier over N', then the functional $Q_{N'}$ such that

$$Q_{N'}(s, x', x'', \varphi) = L_1 R_1(s, x', x'', \varphi), \text{ if } Qy'(L_1(s, y', x'', \varphi) = 0)$$
$$Q_{N'}(s, x', x'', \varphi) = R_1^2(s, x', x'', \varphi), \text{ if } Q^{\cup}y' \neg (L_1(s, y', x'', \varphi) = 0)$$

and $Q_{N'}(s, x', x'', \varphi)\uparrow$ otherwise, embodies Q-quantification.

As far as functions recursive in functionals are concerned, the theory of chapter 13 applies in part since $\mathscr{S}$ is now only a pairing space. While for all functions $\psi \in \mathscr{F}$ the functionals $\check{\psi} = \lambda s\varphi.\psi(s)$ are in $\mathscr{F}_1$, there is no multiplication in $\mathscr{S}$, hence no functionals $\bar{\psi}, Id, Ml$. (Multiplication can sometimes be introduced. Whenever $N = M$, then one may define $\varphi\psi$ as $\lambda sx.\psi(\varphi(s, x), x)$, thus transforming $\mathscr{S}$ into the IOS of example 22.4.)

The central notion of $\mathscr{B}_1$-recursiveness corresponds to $\mathscr{B}'$-recursiveness of chapter 13. Given a subset $\mathscr{B}_1$ of $\mathscr{F}_1$ and a nonempty subset $\mathscr{B}$ of $\mathscr{F}$, a function φ is *$\mathscr{B}_1$-recursive in $\mathscr{B}$* iff the functional $\check{\varphi}$ is recursive in $\check{\mathscr{B}} \cup \mathscr{B}_1$, while a mapping $\Gamma:\mathscr{F} \to \mathscr{F}$ is *$\mathscr{B}_1$-recursive in $\mathscr{B}$* iff the functional

$\Gamma^{\cup} = \lambda s\varphi.\Gamma(\varphi)(s)$ is recursive in $\check{\mathscr{B}} \cup \mathscr{B}_1$. Notions of prime $\mathscr{B}_1$-recursiveness, $\mathscr{B}_1$-primitiveness etc. are also introduced.

Ordinary properties as 13.1–13.7 hold in the present context. Others such as 13.8, 13.9* cannot now be claimed or even formulated. Sparse as the present theory may be, it includes the following key Normal Form Theorem, First Recursion Theorem and Enumeration Theorem. (Given are nonparametrized versions.)

Proposition 28.4. Let $\mathscr{B}_0 \subseteq \mathscr{F}_1$ be as in chapter 7 and let φ be $\mathscr{B}_1$-recursive in $\mathscr{B}$. Then $\varphi = L'(\mu\theta.\Gamma(\theta))$ for a certain Γ such that $\Gamma^{\cup}$ is a member of $\mathscr{F}_1$ strictly polynomial in $\check{\mathscr{B}} \cup \mathscr{B}_0 \cup \langle \mathscr{B}_1 \rangle$. (Therefore, Γ is $\mathscr{B}_1$-primitive in $\mathscr{B}$.)

Proof. Recall that $\mathscr{B}_0$ is a finite set of primitive functionals such that $cl(\mathscr{B}_0/\circ, [\ \ ])$ contains all the recursive functionals. It follows from 7.11, 12.32 that $\check{\varphi}$ is prime recursive in $\check{\mathscr{B}} \cup \mathscr{B}_0 \cup \langle \mathscr{B}_1 \rangle$. Repeating the proof of 13.11 with Id replaced by $\check{\psi}$ for a certain $\psi \in \mathscr{B}$, we get the desired normal form.

A modified version of 13.15 asserts that whenever φ is $\mathscr{B}_1$-recursive in $\mathscr{B}$, then

$$\varphi = L'(R'(\mu\theta.\Gamma((\psi_0, \ldots, \psi_m, \theta, \langle \Gamma_1 \rangle(\theta), \ldots, \langle \Gamma_n \rangle(\theta))))),$$

where $\psi_0, \ldots, \psi_m \in \mathscr{B}$, $\Gamma_1^{\cup}, \ldots, \Gamma_n^{\cup} \in \mathscr{B}_1$ and $\Gamma^{\cup}$ is a strictly primitive member of $\mathscr{F}_1$.

Proposition 28.5. Whenever a mapping $\Gamma: \mathscr{F} \to \mathscr{F}$ is $\mathscr{B}_1$-recursive in $\mathscr{B}$, the function $\mu\theta.\Gamma(\theta)$ exists and is $\mathscr{B}_1$-recursive in $\mathscr{B}$.

Proof. 12.27 gives $(\mu\theta.\Gamma(\theta))^{\vee} = R_1[\Gamma^{\cup} R_1]$.

Systems of equalities are easily solved, allowing for simultaneous recursion.

Proposition 28.6. Let $\mathscr{B}, \mathscr{B}_1$ be finite and let $\mathscr{U}$ stand for the set of all functions $\mathscr{B}_1$-recursive in $\mathscr{B}$. Then there is a $\sigma \in \mathscr{U}$ which is univeral for $\mathscr{U}$, i.e. $\mathscr{U} = \{L'(R'^n(\sigma))/n \in \omega\}$.

Proof. Take a $\psi \in \mathscr{B}$ and a functional Σ recursive in $\check{\mathscr{B}} \cup \mathscr{B}_1$ and universal for all such functionals by 9.18, then take $\check{\sigma} = \Sigma\check{\psi}$. The function σ will do.

As the remarks to 13.18 indicate, the essential assumption above is that $\mathscr{B}_1$ be finite.

The results of chapter 9 concerning universal elements and mappings remain valid with minor modifications due to the fact that L', R' are no longer of the form $\bar{L}, \bar{R}$. For instance, the Second Recursion Theorem asserts that whenever $\sigma \in \mathscr{U}$ is principal universal for $\mathscr{U}$ and $\varphi \in \mathscr{U}$, then there is an n such that $L'(R'^n(\varphi)) = L'(R'^n(\sigma))$. (A function σ is principal universal for $\mathscr{U}$ iff for all $\varphi \in \mathscr{U}$ there is a natural in the sense of chapter 9 primitive recursive functional Ψ such that $\forall n(\bar{n}\check{\varphi} = \bar{n}\Psi\check{\sigma})$.)

The following statement introduces a t-operation St in the space $\mathscr{S}_1$. As a result we have notions of st-*recursiveness* for functionals and $\mathscr{B}_1$, st-*recursiveness* for functions. The general theory of chapter 10 applies to the former, while the remarks following 13.20* concern the latter.

Proposition 28.7. Let $\mathscr{S}_1$ be the IOS of example 28.1 and let M_0, J be the same as in 21.13. Take

$$St(\phi)(J(s,t),\varphi) = \phi(t, \lambda r.\varphi(J(s,r)))$$

and $St(\phi)(s,\varphi)\uparrow$ otherwise. Then St is a t-operation over $\mathscr{F}_1$ satisfying axiom $\mathrm{t}\mu\mathrm{A}_3$.

Proof. Take $\mathscr{L} = M\tilde{}_0$, where $\tilde{s} = \lambda t\varphi.\varphi(J(s,t))$,

$$K_0(J(s,f_1(t)),\varphi) = \varphi(f_1(J(s,t))),$$
$$K_0(J(s,f_2(t)),\varphi) = \varphi(f_2(J(s,t))),$$
$$K_1(J(s,J(t,r)),\varphi) = \varphi(J(J(s,t),r)),$$
$$K_2(J(J(s,t),r),\varphi) = \varphi(J(s,J(t,r)))$$

and $K_0(s,\varphi)$, $K_1(s,\varphi)$, $K_2(s,\varphi)\uparrow$ otherwise.

It follows easily that $\tilde{s}K_0 = (\tilde{s}L_1, \tilde{s}R_1)$, $\tilde{s}\tilde{t}K_1 = J(t,s)\tilde{}$ and $J(t,s)\tilde{}K_2 = \tilde{s}\tilde{t}$ for all $s,t \in M_0$. The equality $\tilde{s}St(\phi) = \phi\tilde{s}$ and the implication

$$\forall\tilde{s}(\tilde{s}\phi \le \tilde{s}\Psi) \Rightarrow St(I_1)\phi \le St(I_1)\Psi$$

follow by the definition of St; hence the axiom (\$) is valid by exercise 10.7. Therefore, St is a storing operation, hence it is a t-operation and satisfies $\mathrm{t}\mu\mathrm{A}_3$ by 10.18, 18.21. The proof is complete.

Now we can briefly describe the approach of Kechris and Moschovakis [1977].

We consider monotonic functionals

$$\phi: M^n \times \mathscr{P}\mathscr{I}_{k_1} \times \cdots \times \mathscr{P}\mathscr{I}_{k_m} \longrightarrow \omega,$$

where $\mathscr{P}\mathscr{I}_k = \{f / f : M^k \longrightarrow \omega\}$, $\omega \subseteq M$. Given a class of functionals $\mathscr{I}$, a functional ϕ is *$\mathscr{I}$-recursive* iff

$$\phi(s_1,\ldots,s_n,f_1,\ldots,f_m) = \Psi^\infty(n_1,\ldots,n_k,s_1,\ldots,s_n,f_1,\ldots,f_m)$$

for certain $\Psi \in \mathscr{I}$, $n_1,\ldots,n_k \in \omega$, where Ψ^∞ is the least functional satisfying the equality

$$\begin{aligned}&\Theta(t_1,\ldots,t_{n+k},f_1,\ldots,f_m)\\ &\quad= \Psi(t_1,\ldots,t_{n+k},\lambda r_1\ldots r_{n+k}.\Theta(r_1,\ldots,r_{n+k},f_1,\ldots,f_m),f_1,\ldots,f_m).\end{aligned}$$

In order to get a nontrivial theory the class $\mathscr{I}$ is assumed to be suitable, i.e. to contain certain specific functionals and to be closed under several operations on functionals. A functional ϕ is said to be recursive in a list of functionals $\Psi_1,\ldots,\Psi_n$ iff ϕ is $\mathscr{I}_0[\Psi_1,\ldots,\Psi_n]$-recursive, where $\mathscr{I}_0[\Psi_1,\ldots,\Psi_n]$ is the smallest suitable class containing $\Psi_1,\ldots,\Psi_n$. Recursion in so called normal lists in studied, including recursion in functionals embodying quantifiers.

While $\mathscr{I}$-recursiveness has its origins in Inductive Definability Theory, it is worth mentioning that the basic notion of inductive relation is in turn described via $\mathscr{I}$-recursiveness in the referred work of Kechris and Moschovakis (the 'boldface' version) and Kolaitis [1978] (the 'lightface' one).

We shall establish no precise characterization of $\mathscr{I}$-recursiveness, noticing only that it is st-recursiveness in the following particular instance of example 28.1.

Proposition 28.8 (Example 28.3). Let $\omega \subseteq M$ and $M^* = \bigcup_n M^n$. Take

$$\begin{aligned}
&\mathscr{F} = \{\varphi/\varphi : \omega \times M^* \dashrightarrow \omega\},\\
&\mathscr{F}_1 = \{\phi/\phi : \omega \times M^* \times \mathscr{F} \dashrightarrow \omega \,\&\, \phi \text{ is monotonic}\},\\
&\phi\Psi = \lambda sx\varphi.\phi(s, x, \lambda ty.\Psi(t, y, \varphi)),\\
&(\phi, \Psi)(2s, x, \varphi) = \phi(s, x, \varphi),\\
&(\phi, \Psi)(2s+1, x, \varphi) = \Psi(s, x, \varphi),
\end{aligned}$$

$I_1 = \lambda sx\varphi.\varphi(s, x)$, $L_1 = \lambda sx\varphi.\varphi(2s, x)$ and $R_1 = \lambda sx\varphi.\varphi(2s+1, x)$. Then $\mathscr{S}_1 = (\mathscr{F}_1, I_1, \Pi_1, L_1, R_1)$ is a $(**)_0$-complete OS. Introduce multiplication $\varphi\psi = \lambda sx.\psi(\varphi(s, x), x)$ in $\mathscr{F}$ and take $(\varphi, \psi)(2s, x) = \varphi(s, x)$, $(\varphi, \psi)(2s+1, x) = \psi(s, x)$, $I = \lambda sx.s$, $L = \lambda sx.2s$, $R = \lambda sx.2s+1$. Then $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ and $\mathscr{S}_1$ are consecutive IOS. (That is, so are $\mathscr{S}$ and the isomorphic copy $\mathscr{S}'$ of $\mathscr{S}_1$ consisting of mappings over $\mathscr{F}$.)

Proof. The IOS $\mathscr{S}, \mathscr{S}_1$ are particular instances respectively of examples 22.4, 28.2 and they are consecutive by 19.11. This completes the proof.

Explicit characterizations of the initial operations of $\mathscr{S}_1$ are obtained by 28.3, observing that $\omega = \bigcup_n f_2^n(f_1(\omega))$. The theory of chapter 13 applies completely since $\mathscr{S}$ is now an IOS.

The following specific functions and functionals may be assumed initial.

$$\begin{aligned}
&Id = \lambda sx\varphi.s, \quad Ml = \lambda sx\varphi.L\varphi R\varphi(s, x), \quad \psi_0 = \lambda sx.R(s)\mathrm{sg}s,\\
&\psi_1 = \lambda sx.\overline{lh(x)}(s), \text{ where } lh(t_1, \dots, t_n) = n,\\
&\psi_2(s, t, x) = L(s), \text{ if } t \in \omega, \text{ otherwise } \psi_2(s, t, x) = R(s),\\
&\Psi_0(s, \varphi) = \varphi(s, s),\ \Psi_0(s, t, x, \varphi) = \varphi(s, s, t, t, x),\\
&\Psi_1(\bar{n}(s), x, \varphi) = \varphi(s, x_1), \text{ if } lh(x) > n
\end{aligned}$$

and x_1 is x with its first component moved to the $n+1$-th place,

$$\begin{aligned}
&\Psi_2(s, t, x, \varphi) = \varphi(t, s, x), \text{ if } t \in \omega,\\
&\Psi_3(s, t, x, \varphi) = \varphi(s, x)
\end{aligned}$$

and $\Psi_1(s, x, \varphi) - \Psi_3(s, x, \varphi)\uparrow$ otherwise.

The operation St is introduced by 28.7, using the pairing function $J((s, x), (t, y)) = (\overline{lh(x)}\bar{s}(t), x, y)$.

EXERCISES TO CHAPTER 28

Exercise 28.1. Let $\mathscr{S}$ be the IOS of example 28.2 and let M_0, J be as in 21.13. Take $St(\phi)(J(s, t), x, \varphi) = \phi(t, x, \lambda ry.\varphi(J(s, r), y))$ and $St(\phi)(s, x, \varphi)\uparrow$ otherwise. Show that St is a t-operation satisfying axiom $\mathrm{t}\mu\mathrm{A}_3$.

Hint. Take a fixed $x_0 \in N_1$ and $J_1((s, x_0), (t, x)) = (J(s, t), x)$ for all $s \in M_0$, $t \in M$, $x \in N_1$. Then the storing operation corresponding to J_1 by 28.7 is exactly St.

Exercise 28.2. Show that the operations St introduced by 28.7 and exercise 28.1 satisfy the assumptions of exercises 10.2–10.5, 10.9 for appropriate functionals K_4–K_6.

Hint. Cf. exercise 21.7.

Remarks. In view of the comments to 10.9, a 'boldface' version of the theory may also be pursued.

Exercise 28.3. Let $\mathscr{S}_1, St$ be as in 28.7 with $N = M$. Using the same pairing function J, introduce an operation St_0 over $\mathscr{F}$ by exercise 21.2. Show that $St(\bar{\varphi}) = \overline{St_0(\varphi)}$ for all φ, where $\bar{\varphi} = \lambda s\psi . \varphi\psi(s)$.

Remark. Therefore, the isomorphism between the IOS $\mathscr{S}$, $\bar{\mathscr{S}}$ agrees with the operation St. A similar assertion holds for the operation St of exercise 28.1.

Exercise 28.4. Let $\mathscr{S}_1, St$ be as in 28.7 (exercise 28.1) and $M_0 = M$. Construct an iterative combinatory space $\mathscr{S}^*$ based on $\mathscr{F}_1$ such that St is exactly the operation of 27.13.

Hint. Use exercise 27.6 (exercise 27.8) with $E = N \cup \{\perp\}$ and $s \leq t$ iff $s = t$ in M.

CHAPTER 29

Higher recursion theory

The notion of Kleene-recursive function with finite type arguments was introduced by Kleene [1959], enabling the recursion theory of such functions to grow considerably in the subsequent period. Here we suggest an alternative approach based on the notion of $\mathscr{B}_1$-recursiveness of chapter 28, hence quite similar to the approach of Kechris and Moschovakis [1977]. The latter work provides a comprehensive introduction to the subject, while this chapter simply marks the place of Higher Recursion Theory from the Algebraic Recursion Theory viewpoint.

Take $T^{(0)} = \omega$, $T^{(j+1)} = \{\alpha^{j+1}/\alpha^{j+1}: T^{(j)} \to \omega\}$ (the members of $T^{(j)}$ are called *j-objects*), $T = \bigcup_j T^{(j)}$ and

$$T^* = \bigcup_n T^n = \{x = (\alpha_1, \dots, \alpha_n)/n \in \omega \,\&\, \alpha_1, \dots, \alpha_n \in T\},$$

assuming that the i-objects in x precede the j-objects whenever $i < j$. Let

$$T^*_m = \{x \in T^*/ar(x) = m\},$$

where $ar(x) = \langle n_0, \dots, n_{j_k} \rangle$, n_i is the number of all i-objects in $x = (\alpha^{j_1}, \dots, \alpha^{j_k})$ for all $i \leq j_k$. Writing p_i for the i-th prime number, the coding function used is $\langle k_0, \dots, k_n \rangle = p_0^{k_0+1} \dots p_n^{k_n+1}$ with inverses $(\langle k_0, \dots, k_n \rangle)_i = k_i$ for $i \leq n$ and $(k)_i \uparrow$ otherwise.

A function $f: T^* \dashrightarrow \omega$ is called *m-ary* if $\mathrm{Dom}\, f \subseteq T^*_m$. Such functions with finite type arguments are called functionals in Kleene [1959], but here the terms 'function' and 'functional' are used in the sense of chapter 28.

Example 29.1. The IOS $\mathscr{S}_1$ of example 28.3 with $M = T$.

Here, one has consecutive IOS $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ and $\mathscr{S}_1 = (\mathscr{F}_1, I_1, \Pi_1, L_1, R_1)$, where

$$\begin{aligned}
&\mathscr{F} = \{\varphi/\varphi: \omega \times T^* \dashrightarrow \omega\},\\
&\varphi\psi = \lambda sx.\psi(\varphi(s,x), x),\\
&(\varphi, \psi)(2s, x) = \varphi(s, x), (\varphi, \psi)(2s+1, x) = \psi(s, x),\\
&I = \lambda sx.s, L = \lambda sx.2s, R = \lambda sx.2s+1,\\
&\mathscr{F}_1 = \{\phi/\phi: \omega \times T^* \times \mathscr{F} \dashrightarrow \omega \,\&\, \phi \text{ is monotonous}\},\\
&\phi\Psi = \lambda sx\varphi.\phi(s, x, \lambda ty.\Psi(t, y, \varphi)),\\
&(\phi, \Psi)(2s, x, \varphi) = \phi(s, x, \varphi), (\phi, \Psi)(2s+1, x, \varphi) = \Psi(s, x, \varphi),\\
&I_1 = \bar{I}, L_1 = \bar{L}, R_1 = \bar{R},
\end{aligned}$$

$\bar{\psi}$ standing for $\lambda sx\varphi.\psi\varphi(s,x)$. Other familiar notations are $\bar{n}$ for LR^n or $L_1R_1^n, \check{\psi} = \lambda sx\varphi.\psi(s,x)$. We also recall that

$$\langle \phi \rangle (\bar{n}(s), x, \varphi) = \phi(s, x, \lambda ty.\varphi(\bar{n}(t), y)),$$
$$\lambda sx.[\phi](s, x, \varphi) = \mu\theta.(\varphi, \lambda sx.\phi(s, x, \theta)).$$

The following functions and functionals are to be assumed initial.

$$Id = \check{I}, Ml = \lambda sx\varphi.L\varphi R\varphi(s,x),$$
$$\psi_0 = \lambda sx.R(s)\mathrm{sg}s,$$
$$\psi_1 = \lambda sx.ar(x)(s),$$
$$\psi_2(s, t, \alpha^1, x) = \alpha^1(t)$$

(and $\psi_2(s,x)\uparrow$ otherwise; this will often be omitted below),

$$\Psi_0 = \lambda sx\varphi.\varphi(s, s, x), \Psi_1(\overline{kj}(s), x, \varphi) = \varphi(x_1),$$

where x_1 is (s, x) with its first j-object moved to the $k+1$-th place,

$$\Psi_2(\bar{j}(s), \alpha^j, x, \varphi) = \alpha^j(\lambda\alpha^{j-2}.\varphi(s, \alpha^{j-2}, \alpha^j, x)), \text{ if } j \geq 2,$$

$\Psi_3(\bar{j}(s), x, \varphi) = \varphi(s, x_1)$, where x_1 is x with its first j-object dropped.

Instead of ψ_2, Ψ_2 one may take the 'more powerful' ψ_2^+, Ψ_2^+, where

$$\psi_2^+(\bar{j}(s), \alpha^j, \alpha^{j+1}, x) = \alpha^{j+1}(\alpha^j),$$
$$\Psi_2^+(\bar{j}(s), x, \varphi) = L\varphi(s, \lambda\alpha^j.R\varphi(s, \alpha^j, x), x).$$

The corresponding two basic sets $\mathscr{B}_1$,

$$\mathscr{K}_1 = \{\check{\psi}_0, \check{\psi}_1, \check{\psi}_2, Id, Ml, \Psi_0, \Psi_1, \Psi_2, \Psi_3\}$$

and

$$\mathscr{K}_1^+ = \{\check{\psi}_0, \check{\psi}_1, \check{\psi}_2^+, Id, Ml, \Psi_0, \Psi_1, \Psi_2^+, \Psi_3\}$$

give two notions of computability in higher types, *$\mathscr{K}_1$-recursiveness* and *$\mathscr{K}_1^+$-recursiveness*. We recall that φ is $\mathscr{K}_1$-recursive in $\mathscr{B} \subseteq \mathscr{F}$ iff $\check{\varphi}$ is recursive in $\check{\mathscr{B}} \cup \mathscr{K}_1$ in the sense of $\mathscr{S}_1$. Functions $f: T^* \dashrightarrow \omega$ are represented by members $f^* = \lambda sx.f(x)$ of $\mathscr{F}$, hence f is $\mathscr{K}_1$-recursive in $g_0, \ldots, g_l$ iff f^* is $\mathscr{K}_1$-recursive in $g_0^*, \ldots, g_l^*$. Similarly for $\mathscr{K}_1^+$-recursiveness.

The original relative Kleene-recursiveness is proved by Kleene [1959] to be nontransitive, hence it can be equivalent to $\mathscr{B}_1$-recursiveness for no $\mathscr{B}_1$. The characterization established below shows that absolute Kleene-recursiveness is nevertheless equivalent to $\mathscr{K}_1$-recursiveness. (As proved by Kechris and Moschovakis [1977], it is also equivalent to $\mathscr{I}$-recursiveness.) The exercises to this chapter show that relative Kleene-recursiveness is wider than relative $\mathscr{K}_1$-recursiveness and narrower than relative $\mathscr{K}_1^+$-recursiveness.

It is now time to formulate the definition of m-ary function f Kleene-recursive in a list of functions $g_0, \ldots, g_l$, where g_i is m_i-ary, $i \leq l$. (The following definition is not exactly the original one but a slightly modified equivalent version from Kechris and Moschovakis [1977].) For the sake of simplicity assume that $l = 1$ and say, $g_0: \omega \times T^{(2)} \dashrightarrow \omega$, $g_1: T^{(1)} \times T^{(3)} \dashrightarrow \omega$, i.e. $m_0 = \langle 1, 0, 1 \rangle$ and $m_1 = \langle 0, 1, 0, 1 \rangle$. A mapping $\Omega: \mathscr{F} \to \mathscr{F}$ is introduced via the following clauses for $\Omega(\varphi)$.

0_0. $\Omega(\varphi)(\langle 0, ar(n,x), 0, e_1\rangle, n, x) = g_0(n, \lambda\alpha^1.\varphi(e_1, \alpha^1, x))$.
0_1. $\Omega(\varphi)(\langle 0, ar(x), 1, e_1, e_2\rangle, x) = g_1(\lambda\alpha^0.\varphi(e_1, \alpha^0, x), \lambda\alpha^2.\varphi(e_2, \alpha^2, x))$.
1. $\Omega(\varphi)(\langle 1, ar(t,x)\rangle, t, x) = t+1$.
2. $\Omega(\varphi)(\langle 2, ar(x), t\rangle, x) = t$.
3. $\Omega(\varphi)(\langle 3, ar(t,x)\rangle, t, x) = t$.
4. $\Omega(\varphi)(\langle 4, ar(x), e_1, e_2\rangle, x) = \varphi(e_2, \varphi(e_1, x), x)$.
5. $\Omega(\varphi)(\langle 5, ar(r,t,l,m,n,x)\rangle, r,t,l,m,n,x) = \begin{cases} p^t, & \text{if } m = n, \\ lt, & m \neq n. \end{cases}$
6. $\Omega(\varphi)(\langle 6, ar(x), j, k, e_1\rangle, x) = \varphi(e_1, x_1)$, where x has at least $k+1$ j-objects and x_1 is obtained from x by moving the first j-object to the $k+1$-th place.
7. $\Omega(\varphi)(\langle 7, ar(t, \alpha^1, x)\rangle, t, \alpha^1, x) = \alpha^1(t)$.
8. $\Omega(\varphi)(\langle 8, ar(\alpha^j, x), j, e_1\rangle, \alpha^j, x) = \alpha^j(\lambda\alpha^{j-2}.\varphi(e_1, \alpha^j, \alpha^{j-2}, x))$, if $j \geq 2$.
9. $\Omega(\varphi)(\langle g, ar(t,x,y), ar(x)\rangle, t, x, y) = \varphi(t, x)$.

Otherwise $\Omega(\varphi)(s,x)\uparrow$.

The mapping Ω is monotonic; hence $\sigma = \mu\theta.\Omega(\theta)$ exists. Moreover, it follows from the proof of 18.1 that $\sigma = \sigma_\zeta$ for a certain ordinal ζ, where $\sigma_\xi = \sup_{\eta<\xi}\Omega(\sigma_\eta)$ for all ξ.

A function f is *Kleene-recursive in* $g_0, \ldots, g_l$ iff $f = \lambda x.\sigma(e,x)$ with a certain index (natural number) e. Traditionally, $\{e\}(x)$ is written for $\sigma(e,x)$. We shall also write $\{e\}_\xi(x)$, $\{e\}_{<\xi}(x)$ respectively for $\sigma_\xi(e,x)$ $(\sup_{\eta<\xi}\sigma_\eta)$ (e,x). If $\sigma(e,x)\downarrow$, then (e,x) has an *ordinal of computation*, $|e,x| = \min\{\xi/\sigma_\xi(e,x)\downarrow\}$.

The initial functions $g_0, \ldots, g_l$ may include $j+2$-objects ${}^{j+2}F_Q$, where Q is a monotonic quantifier over $T^{(j)}$ and

$$ {}^{j+2}F_Q(\alpha^{j+1}) = \begin{cases} 0, & \text{if } Q\alpha^j(\alpha^{j+1}(\alpha^j) = 0), \\ 1, & \text{if } Q^\smile\alpha^j(\alpha^{j+1}(\alpha^j) \neq 0). \end{cases} $$

Kleene-recursiveness in lists $\alpha^{j+2}, {}^{j+2}E$ has been studied extensively, ${}^{j+2}E$ standing for ${}^{j+2}F_\exists$.

Let $f: T^* \longrightarrow \omega$ be m-ary, writing explicitly $f(z, \alpha^{j_1+1}, \ldots, \alpha^{j_k+1})$ with z ranging over $\omega^{(m)_0}$. Bearing clause 0 in mind, assign a functional $\tilde{f} \in \mathscr{F}_1$ to f such that

$$ \tilde{f}(s, z, x, \varphi) = f(z, \lambda\alpha^{j_1}.\overline{0}\varphi(s, \alpha^{j_1}, x), \ldots, \lambda\alpha^{j_k}.\overline{k-1}\,\varphi(s, \alpha^{j_k}, x)). $$

The main result of this chapter states that a function f is Kleene-recursive in $g_0, \ldots, g_l$ iff f is $\mathscr{K}_1 \cup \{\widetilde{g_0}, \ldots, \widetilde{g_l}\}$-recursive, i.e. iff $f^{*\smile}$ is recursive in $\mathscr{K}_1 \cup \{\widetilde{g_0}, \ldots, \widetilde{g_l}\}$. In order to prove the 'only if'-part of the equivalence, some $\mathscr{K}_1$-recursive functions and functionals will be constructed by using the following auxiliary statement.

Proposition 29.1. For every partial recursive function $f: \omega^n \longrightarrow \omega$ there is a $\mathscr{K}_1$-recursive function $\varphi \in \mathscr{F}$ such that $\varphi(s_1, \ldots, s_n, x) = f(s_1, \ldots, s_n)$ for all x and $\varphi(s,x)\uparrow$ otherwise.

Proof. Take $I^n_k(s_1, \ldots, s_n, x) = s_k$, and $I^n_k(s,x)\uparrow$ whenever $(ar(s,x))_0 < n$, $1 \leq k \leq n$. It follows that $I^n_1 L(s,x) = 2s$, provided $(ar(s,x))_0 \geq n$, and similarly for $I^n_1 R$, $I^n_1\psi_0$. The characterization established by 22.7 shows that there is a function φ which is recursive in ψ_0, I^n_k, $1 \leq k \leq n$ in the sense of $\mathscr{S}$

and corresponds to f as required. The equalities $I_1^{n\smile} = \overline{(n-1\bar{0}\Psi_1)}^n Id$ and $I_k^{n\smile} = \overline{(n-1\bar{0}\Psi_1)}^{k-1} Id$ for $1 < k \leq n$ imply that the functions I_k^n are $\mathscr{K}_1$-recursive, hence so is φ. The proof is complete.

Proposition 29.2. If f is Kleene-recursive in $g_0, \ldots, g_l$, then f is $\mathscr{K}_1 \cup \{\widetilde{g_0}, \ldots, \widetilde{g_l}\}$-recursive.

Proof. The function f has a Kleene index e, hence $f^* = \chi\sigma$, where $\chi = \lambda sx.e$ is $\mathscr{K}_1$-recursive by 29.1. But is the universal function σ $\mathscr{K}_1 \cup \{\widetilde{g_0}, \ldots, \widetilde{g_l}\}$-recursive? Noting that $\sigma = \mu\theta.\Omega(\theta)$, it suffices by 13.16 to prove that Ω is a $\mathscr{K}_1 \cup \{\widetilde{g_0}, \ldots, \widetilde{g_l}\}$-recursive mapping, i.e. $\Omega^{\smile} = \lambda sx\varphi.\Omega(\varphi)(s,x)$ is a functional recursive in $\mathscr{K}_1 \cup \{\widetilde{g_0}, \ldots, \widetilde{g_l}\}$.

There are by 29.1 $\mathscr{K}_1$-recursive functions $\varphi_0, \ldots, \varphi_6$ such that

$$\varphi_0(s,n,x) = n+1,\ \varphi_1 = I_2^2,\ \varphi_i = \lambda sx.(s)_i,\ i = 2,3,4,$$

$$\varphi_5(s,r,t,l,m,n,x) = \begin{cases} r^t, & \text{if } m = n, \\ lt, & \text{if } m \neq n, \end{cases}$$

$\varphi_6(s,x) = \overline{(s)_0}(s)$, if $(s)_0 = 1, \ldots, 9$ and $lh(s)$ equals respectively $2,3,2,4,2,5,2,4,3$; $\varphi_6(s,x) = \overline{(s)_2}\,\overline{(s)_0}(s)$, if $(s)_0 = 0$ and either $(s)_2 = 0$, $lh(s) = 4$, or $(s)_2 = 1$, $lh(s) = 5$, where $lh(s) = n > 0$ means that $s = \langle (s)_0, \ldots, (s)_{n-1} \rangle$.

The function $\rho = \lambda sx.\overline{(s)_1}(s)$ is $\mathscr{K}_1$-recursive by 29.1, and there is by 8.1 a recursive χ such that $\bar{s}\bar{s}\chi = \bar{0}$ and $\bar{s}\bar{t}\chi = \bar{1}$ whenever $s \neq t$. Therefore, the function $\varphi_7 = \psi_1 \langle \rho \rangle \chi(I, O)$ is $\mathscr{K}_1$-recursive and $\varphi_7(s,x) = s$, provided $(s)_1 = ar(x)$.

Take $\phi_0 = \bar{1}\bar{0}\Psi_1\bar{0}\Psi_3$. Then $\phi_0(s,n,x,\varphi) = \varphi(n,x)$.

Take $\phi_1 = \bar{1}\bar{0}\Psi_1$. Then $\phi_1(s,n,x,\varphi) = \varphi(n,s,x)$ and

$$\Psi_0 Ml(\phi_0\overline{\varphi_2}, \phi_1\overline{\varphi_3})(s,x,\varphi) = \varphi((s)_3, \varphi((s)_2,x),x).$$

The function $\rho = \lambda sx.\overline{(s)_3(s)_2}((s)_4)$ is $\mathscr{K}_1$-recursive. Taking $\phi_2 = \bar{\rho}(\langle \phi_1 \rangle R_1 L_1, R_1)\Psi_1$, we get that $\phi_2(s,x,\varphi) = \varphi((s)_4, x_1)$, if $(s)_2 = j$, $(s)_3 = k$, $(ar(x))_j > k$ and x_1 is x with its first j-object moved to the $k+1$-th place.

The function $\rho = \lambda sx.\overline{(s)_2}((s)_3)$ is $\mathscr{K}_1$-recursive. Take $\phi_3 = \bar{\rho}\Psi_2$. Then

$$\phi_3(s,\alpha^j,x,\varphi) = \alpha^j(\lambda\alpha^{j-2}.\varphi((s)_3,\alpha^{j-2},\alpha^j,x)),$$

provided $(s)_2 = j \geq 2$.

Using ψ_1, Ψ_1, Ψ_3, we construct finally a functional ϕ_4 recursive in $\mathscr{K}_1$ such that $\phi_4(s,n,x,y,\varphi) = \varphi(n,x)$, provided $(s)_2 = ar(x)$.

It is now evidently the case that

$$\begin{aligned}\Omega^{\smile} = \overline{\varphi_7}\,\overline{\varphi_6}((\widetilde{g_0}(\overline{\varphi_3}, O_1), \widetilde{g_1}(\overline{\varphi_3}, \overline{\varphi_4}, O_1)), \check{\varphi}_0, \check{\varphi}_2, \check{\varphi}_1, \Psi_0 Ml(\phi_0\overline{\varphi_2}, \phi_1\overline{\varphi_3}),\\ \check{\varphi}_5, \phi_2, \check{\psi}_2, \phi_3, \phi_4, O_1),\end{aligned}$$

hence $\Omega^{\smile}$ is recursive in $\mathscr{K}_1 \cup \{\widetilde{g_0}, \ldots, \widetilde{g_l}\}$. The proof is complete.

Proposition 29.3. If an m-ary function f is $\mathscr{K}_1 \cup \{\widetilde{g_0}, \ldots, \widetilde{g_l}\}$-recursive, then f is Kleene-recursive in $g_0, \ldots, g_l$.

Proof. We are going to show that for every functional ϕ recursive in $\mathscr{K}_1 \cup \{\widetilde{g_0}, \ldots, \widetilde{g_l}\}$ there is an index e_1 such that $\lambda n.\{e_1\}(n)$ is a primitive recursive (hence total) function and

$$\phi(s, x, \lambda ty.\{\{e\}(ar(y))\}(t, y)) = \{\{e_1\}(ar(x))\}(\langle e, s\rangle, x)$$

for all e, s, x. This will easily imply the desired conclusion. Indeed, take e_1 to correspond to $f^{*\smile}$. Then

$$f(x) = f^{*\smile}(0, x, \lambda ty.\{\{0\}(ar(y))\}(t, y)) = \{\{e_1\}(m)\}(\langle 0, 0\rangle, x)$$

for all x; hence f is Kleene-recursive in $g_0, \ldots, g_l$.

Using induction on the construction of ϕ, we shall find a corresponding index e_1 and prove in addition that whenever $\{\{e_1\}(ar(x))\}_\xi(\langle e, s\rangle, x) = u$, then $\phi(s, x, \lambda ty.\{\{e\}(ar(y))\}_{<\xi}(t, y)) = u$ for all ξ.

1. Let $\phi = \widetilde{g_0}$. Then

$$\begin{aligned}
&\widetilde{g_0}(s, n, x, \lambda ty.\{\{e\}(ar(y))\}(t, y))\\
&\quad = g_0(n, \lambda\alpha^1.\{\{e\}(ar(\alpha^1, x))\}(2s, \alpha^1, x))\\
&\quad = \{\langle 0, 4ar(x), 0, \{e\}(3ar(x))\rangle\}(n, 2s, x)\\
&\quad = \{h_1(ar(x), e)\}(n, 2s, x)\\
&\quad = \{\langle 6, 4ar(x), 0, 1, h_1(ar(x), e)\rangle\}(2s, n, x)\\
&\quad = \{h_2(ar(x), e)\}(\{\langle 2, 2ar(x), 2s\rangle\}(n, x), n, x)\\
&\quad = \{\langle 4, 2ar(x), \langle 2, 2ar(x), 2s\rangle, h_2(ar(x), e)\rangle\}(n, x)\\
&\quad = \{h_3(ar(x), e, s)\}(n, x)\\
&\quad = \{\langle 9, 8ar(x), 2ar(x)\rangle\}(h_3(ar(x), e, s), n, x, \langle e, s\rangle)\\
&\quad = \{h_4(ar(x))\}(h_3(ar(x), e, s), \langle e, s\rangle, n, x)\\
&\quad = \{h_4(ar(x))\}(\{h_5(ar(x))\}(\langle e, s\rangle, n, x), \langle e, s\rangle, n, x)\\
&\quad = \{\langle 4, 4ar(x), h_5(ar(x)), h_4(ar(x))\rangle\}(\langle e, s\rangle, n, x)\\
&\quad = \{\{e_1\}(ar(x))\}(\langle e, s\rangle, n, x)
\end{aligned}$$

with functions h_1, h_2, h_3 Kleene-recursive in $g_0, \ldots, g_l$, primitive recursive functions h_4, h_5 and index e_1 chosen quite obviously, using the fact that all partial recursive functions are Kleene-recursive in $g_0, \ldots, g_l$ (Kleene [1959]). For instance,

$$h_4 = \lambda s.\langle 6, 8s, 0, 1, \langle 6, 8s, 0, (s)_0 + 2, \langle 9, 8s, 2s\rangle\rangle\rangle.$$

The above transformations increase the ordinal of computation. Supposing $\{\{e_1\}(ar(x))\}_\xi(\langle e, s\rangle, n, x) = u$, one gets

$$\{\langle 4, 4ar(x), h_5(ar(x)), h_4(ar(x))\rangle\}_\xi(\langle e, s\rangle, n, x) = u,$$

hence

$$\{h_4(ar(x))\}_{<\xi}(\{h_5(ar(x))\}_{<\xi}(\langle e, s\rangle, n, x), \langle e, s\rangle, n, x) = u$$

etc. Finally, $g_0(n, \lambda\alpha^1.\{\{e\}(ar(\alpha^1, x))\}_{<\xi}(2s, \alpha^1, x)) = u$ gives $\widetilde{g_0}(s, n, x, \lambda ty.\{\{e\}(ar(y))\}_{<\xi}(t, y)) = u$.

2. The functionals $\tilde{g}_1, L_1, A_1 = (R_1, R_1L_1)$, Id, Ml, $\breve{\psi}_0, \breve{\psi}_1, \breve{\psi}_2, \Psi_0, \Psi_1, \Psi_2$

and Ψ_3 are treated similarly; R_1 is replaced by A_1 to eliminate the pairing operation by 6.9.

It remains to show that the operations $\circ, \langle\quad\rangle, [\quad]$ of $\mathscr{S}_1$ preserve the property in question.

3. The case of multiplication. Assume that e_1, e_2 correspond respectively to ϕ, Ψ. There is a primitive recursive function h such that $\{\{e_2\}(ar(x))\}(\langle e, s\rangle, x) = \{\{h(e)\}(ar(x))\}(s, x)$ and h increases the ordinal of computation, i.e. $|\{e_2\}(ar(x)), \langle e, s\rangle, x| < |\{h(e)\}(ar(x)), s, x|$ whenever $\{\{e_2\}(ar(x))\}(\langle e, s\rangle, x)\downarrow$. Therefore,

$$\begin{aligned}
&\phi\Psi(s, x, \lambda ty.\{\{e\}(ar(y))\}(t, y))\\
&\quad = \phi(s, x, \lambda rz.\Psi(r, z, \lambda ty.\{\{e\}(ar(y))\}(t, y)))\\
&\quad = \phi(s, x, \lambda rz.\{\{e_2\}(ar(z))\}(\langle e, r\rangle, z))\\
&\quad = \phi(s, x, \lambda rz.\{\{h(e)\}(ar(z))\}(r, z))\\
&\quad = \{\{e_1\}(ar(x))\}(\langle h(e), s\rangle, x)\\
&\quad = \{\{e_3\}(ar(x))\}(\langle e, s\rangle, x)
\end{aligned}$$

for an appropriate e_3.

Suppose that $\{\{e_3\}(ar(x))\}_\xi(\langle e, s\rangle, x) = u$. Then $\{\{e_1\}(ar(x))\}_{<\xi}(\langle h(e), s\rangle, x) = u$; hence

$$\phi(s, x, \lambda rz.\{\{h(e)\}(ar(z))\}_{<\xi}(r, z)) = u$$

by the inductive assumption for ϕ. It follows that

$$\phi(s, x, \lambda rz.\{\{e_2\}(ar(z))\}_{<\xi}(\langle e, r\rangle, z)) = u;$$

hence

$$\phi(s, x, \lambda rz.\Psi(r, z, \lambda ty.\{\{e\}(ar(y))\}_{<\xi}(t, y))) = u$$

by the induction assumption for Ψ and the monotonicity of ϕ. Therefore,

$$\phi\Psi(s, x, \lambda ty.\{\{e\}(ar(y))\}_{<\xi}(t, y)) = u,$$

hence e_3 corresponding to $\phi\Psi$.

4. The case of translation. Assume that e_1 corresponds to ϕ. Recalling that there are primitive recursive functions h_1, h_2 such that $s = \overline{h_1(s)}(h_2(s))$ for all s, it follows that

$$\begin{aligned}
&\langle\phi\rangle(s, x, \lambda ty.\{\{e\}(ar(y))\}(t, y)\\
&\quad = \langle\phi\rangle(\overline{h_1(s)}(h_2(s)), x, \lambda ty.\{\{e\}(ar(y))\}(t, y))\\
&\quad = \phi(h_2(s), x, \lambda ty.\{\{e\}(ar(y))\}(\overline{h_1(s)}(t), y))\\
&\quad = \phi(h_2(s), x, \lambda ty.\{\{h_3(e, s)\}(ar(y))\}(t, y))\\
&\quad = \{\{e_1\}(ar(x))\}(\langle h_3(e, s), h_2(s)\rangle, x)\\
&\quad = \{\{e_2\}(ar(x))\}(\langle e, s\rangle, x)
\end{aligned}$$

for a certain e_2 and a primitive recursive function h_3 which increase the ordinal of computation. Proceeding as in the case of multiplication, one shows that e_2 corresponds to $\langle\phi\rangle$.

5. The case of iteration. Assume that e_1 corresponds to ϕ. In view of the fact that $\Sigma = [\phi]$ satisfies the equality

$$\Sigma(s, x, \varphi) = \begin{cases} \varphi(s/2, x), & \text{if } s \text{ is even,} \\ \phi((s-1)/2, x), \lambda ty.\Sigma(t, y, \varphi), & \text{if } s \text{ is odd,} \end{cases}$$

take primitive recursive functions h_1, h_2 such that

$$\{\{h_1(e_2, e)\}(ar(x))\}(s, x) = \{\{e_2\}(ar(x))\}(\langle e, s\rangle, x),$$

$$\{\{h_2(e_2)\}(ar(x))\}(s, x) = \begin{cases} \{\{(s)_0\}(ar(x))\}((s)_1/2, x), & \text{if } (s)_1 \text{ is even.} \\ \{\{e_1\}(ar(x))\}(\langle h_1(e_2, (s)_0), ((s)_1 - 1)/2\rangle, x), & \text{if } (s)_1 \text{ is odd} \end{cases}$$

for all e_2, e, s, x and h_1, h_2 increase the ordinal of computation. There is by the (Second) Recursion Theorem for Kleene-recursiveness an index e_2 such that $\{e_2\}(n) = \{h_2(e_2)\}(n)$ for all n.

Writing ρ, τ for $\lambda sx.\{\{e\}(ar(x))\}(s, x)$, $\lambda sx.\{\{h_1(e_2, e)\}(ar(x))\}(s, x)$ respectively, one gets

$$\begin{aligned}(\check{\rho}, \phi\check{\tau})(s, x, \varphi) &= \begin{cases} \{\{e\}(ar(x))\}(s/2, x), & \text{if } s \text{ is even,} \\ \{\{e_1\}(ar(x))\}(\langle h_1(e_2, e), (s-1)/2\rangle, x), & \text{if } s \text{ is odd} \end{cases} \\ &= \{\{h_2(e_2)\}(ar(x))\}(\langle e, s\rangle, x) \\ &= \{\{e_2\}(ar(x))\}(\langle e, s\rangle, x) = \check{\tau}(s, x, \varphi)\end{aligned}$$

for all s, x, φ, hence $(\check{\rho}, \phi\check{\tau}) = \check{\tau}$, which implies $\Sigma\check{\rho} \leq \check{\tau}$ by (££). However, the last inequality says precisely that whenever $\Sigma(s, x, \lambda ty.\{\{e\}(ar(y))\}(t, y)) = u$, then $\{\{e_2\}(ar(x))\}(\langle e, s\rangle, x) = u$.

Conversely, a transfinite induction on ξ shows that whenever $\{\{e_2\}(ar(x))\}_\xi(\langle e, s\rangle, x) = u$, then $\Sigma(s, x, \lambda ty.\{\{e\}(ar(y))\}_{<\xi}(t, y)) = u$. Suppose this is true for all $\eta < \xi$ and $\{\{e_2\}(ar(x))\}_\xi(\langle e, s\rangle, x) = u$. Then

$$\{\{h_2(e_2)\}(ar(x))\}_\xi(\langle e, s\rangle, x) = u.$$

If s is even, then $\{\{e\}(ar(x))\}_{<\xi}(s/2, x) = u$, hence

$$\Sigma(s, x, \lambda ty.\{\{e\}(ar(y))\}_{<\xi}(t, y)) = u.$$

Suppose that s is odd. Then

$$\{\{e_1\}(ar(x))\}_{<\xi}\left(\left\langle h_1(e_2, e), \frac{s-1}{2}\right\rangle, x\right) = u$$

hence

$$\phi\left(\frac{s-1}{2}, x, \lambda rz.\{\{h_1(e_2, e)\}(ar(z))\}_{<\xi}(r, z)\right) = u.$$

Therefore

$$\phi\left(\frac{s-1}{2}, x, \lambda rz.\{\{e_2\}(ar(z))\}_{<\xi}(\langle e, r\rangle, z)\right) = u,$$

which implies by the inductive hypothesis for ξ and the monotonicity of ϕ that

$$\phi\left(\frac{s-1}{2}, x, \lambda rz.\Sigma(r, z, \lambda ty.\{\{e\}(ar(y))\}_{<\xi}(t, y))\right) = u;$$

hence $\Sigma(s, x, \lambda ty.\{\{e\}(ar(y))\}_{<\xi}(t, y)) = u$. Therefore, e_2 corresponds to Σ. The proof is complete.

Proposition 29.4 (Kleene-Recursiveness Theorem). Let f be a m-ary function. Then f is Kleene-recursive in $g_0, \ldots, g_l$ iff f is $\mathscr{K}_1 \cup \{\tilde{g}_0, \ldots, \tilde{g}_l\}$-recursive. In particular, f is Kleene-recursive iff f is $\mathscr{K}_1$-recursive.

Follows by 29.2, 29.3.

The notions of relative $\mathscr{K}_1$-recursiveness and relative $\mathscr{K}_1^+$-recursiveness can be introduced by duly modifying the original definition of Kleene. Namely, consider the following alternative clauses which also originate in Kleene [1959].

0_i^-. $\Omega(\varphi)(\langle 0, ar(x), i\rangle, x) = g_i(x)$.
7^+. $\Omega(\varphi)(\langle 7, ar(\alpha^j, \alpha^{j+1}, x), j\rangle, \alpha^j, \alpha^{j+1}, x) = \alpha^{j+1}(\alpha^j)$.
8^+. $\Omega(\varphi)(\langle 8, ar(x), j, e_1, e_2\rangle, x) = \varphi(e_2, \lambda\alpha^j.\varphi(e_1, \alpha^j, x), x)$.

Let us call Kleene-recursiveness and Kleene$^+$-recursiveness the notions obtained by replacing clause 0 by clause 0^-, respectively clauses 0, 7, 8 by clauses $0^-, 7^+, 8^+$.

Proposition 29.5. Let f be a m-ary function. Then f is Kleene$^-$-recursive in $g_0, \ldots, g_l$ iff it is $\mathscr{K}_1$ -recursive in $g_0, \ldots, g_l$.

Proposition 29.6. Let f be a m-ary function. Then f is Kleene$^+$-recursive in $g_0, \ldots, g_l$ iff it is $\mathscr{K}_1^+$-recursive in $g_0, \ldots, g_l$.

The proofs of 29.5, 29.6 closely follow that of 29.4.

The notion of Kleene-recursiveness can also be formulated via representability in hierarchies of IOS. This is one of the earlier results of the IOS-theory, analogous to a similar one of Platek [1966]. Such representability does not however provide an adequate setting for recursion in higher types, in which we agree with Feferman [1977].

EXERCISES TO CHAPTER 29

Exercise 29.1. Show that ψ_2 is recursive in ψ_2^+.

Exercise 29.2. Show that Ψ_2 is recursive in $\mathscr{K}_1^+$.

Hint. $\Psi_2 = (O_1, O_1, D_1\langle\Psi_2^+\rangle C_1(R_1\breve{\psi}_2^+, [I_1]))$, where C_1, D_1 are the elements of 6.35, 6.41.

Exercise 29.3. Show that ψ_2^+ is $\mathscr{K}_1$-recursive.

Hint. $\psi_2^+ = R_1[(\breve{\psi}_2, D_1\langle R_1\Psi_2\rangle R_1)]$.

Exercise 29.4. Show that f is $\mathscr{K}_1 \cup \{\breve{f}\}$-recursive.

Hint. If f is m-ary, $f(z, \alpha^{j_1+1}, \ldots, \alpha^{j_k+1})$ and $j_1 < j_2 < \cdots < j_k$, then $f^{*\vee} = \overline{\breve{\psi}}_1(O_1, I_1)^m(\breve{f}(\bar{j}_1, \ldots, \bar{j}_k, O_1)\breve{\psi}_2^+, O_1)$. If $j_n = j_{n+1}$ for some n, then Ψ_1 is to be used. Suppose for instance that $m = \langle(m)_0, 2\rangle$, i.e. $k = 2$ and $j_1 = j_2 = 0$. Then $f^{*\vee} = \overline{\psi_1}(O_1, I_1)^m(\breve{f}(\bar{0}, \bar{1}\bar{0}\Psi_1\bar{0}, O_1)\breve{\psi}_2^+, O_1)$.

Exercise 29.5. Show that $\bar{f}$ is recursive in $\{f^{*\check{}}\} \cup \mathscr{K}_1^+$.

Remark. It follows from 29.4 and exercises 29.1, 29.2, 29.4 and 29.5 that relative $\mathscr{K}_1$-recursiveness implies relative Kleene-recursiveness, while the latter implies relative $\mathscr{K}_1^+$-recursiveness.

Exercise 29.6. Show that Ψ_2^+ is not recursive in $\mathscr{K}_1$.

Hint. Take $\omega^* = \bigcup_n \omega^n$, $\rho = I \restriction \omega^*$, $\tau = I \restriction (T^* \backslash \omega^*)$ and prove that whenever ϕ is recursive in $\mathscr{K}_1$, then $\bar{\rho}\phi = \bar{\rho}\phi\bar{\rho}$. This equality fails for Ψ_2^+ since $\bar{\rho}\Psi_2^+ \check{\tau} \neq \bar{\rho}\Psi_2^+ \bar{\rho}\check{\tau}$.

Exercise 29.7. Prove that all $\mathscr{K}_1$-recursive number functions are partial recursive.

Hint. Assign to each $\varphi \in \mathscr{F}$ a $\varphi^*: \omega \dashrightarrow \omega$ such that $\varphi^*(\overline{s_n} \ldots \overline{s_1}\bar{n}(s_0)) = \varphi(s_0, \ldots, s_n)$. Show that for every ϕ recursive in $\mathscr{K}_1$ there is a μ-recursive mapping Γ such that if $\check{\psi} = \phi\check{\varphi}$, then $\psi^* = \Gamma(\varphi^*)$.

Remark. A modification of 29.1 shows that conversely, all partial recursive functions are $\mathscr{K}_1$-recursive.

Exercise 29.8. Construct a function f such that $\bar{f}$ is not recursive in $\{f^{*\check{}}\} \cup \mathscr{K}_1$.

Hint. Adapting a counterexample of Kleene [1963], take a partial recursive function $f_0: \omega \dashrightarrow \{0, 1\}$ which has no general recursive extension. There is (by 9.3) a binary primitive recursive function h such that $f_0(s) = 0$ iff $\exists t(h(s,t) = 0)$. Take $g(s, t, \alpha^2) = 0$, if $h(s,t) > 0$, and $g(s, t, \alpha^2)\uparrow$ otherwise, then take $f = \lambda s\alpha^2 . \alpha^2(\lambda t . g(s, t, \alpha^2))$. By way of contradiction suppose that $\bar{f}$ is recursive in $\{f^{*\check{}}\} \cup \mathscr{K}_1$. Then $\bar{f}$ is recursive in $\mathscr{K}_1$ since both g and f are $\mathscr{K}_1$-recursive. Let $\varphi = \lambda s\alpha^1 . 1$ and $\check{\chi} = \Psi_0 \bar{f} \check{\varphi}$. Then χ is $\mathscr{K}_1$-recursive, hence partial recursive by exercise 29.7. Show that $f_0 \cup \chi$ is a general recursive extension of f_0, a contradiction.

Exercise 29.9. Using the above counterexample, show that relative Kleene-recursiveness is not transitive.

Hint. χ is Kleene-recursive in f which is Kleene-recursive, while χ is not partial recursive, hence not Kleene-recursive.

Remark. Kleene-recursiveness also fails to support a First Recursion Theorem. (Cf. Platek [1966]. Kleene has recently revisited the subject, assuming such a theorem in his new definition.) Proposition 29.4 and exercises 29.4, 29.8 show that these peculiarities are due to the privileged status of the initial functions as compared with other functions Kleene-recursive in the initial ones.

Exercise 29.10. Construct a $\mathscr{K}_1^+$-recursive number function which is not partial recursive.

Hint. Consider the function χ constructed in the hint to exercise 29.8.

Exercise 29.11. Let $\rho(\bar{j}(s), \alpha^{j+1}, x) = 1$, if $\forall \alpha^j(\alpha^{j+1}(\alpha^j) \neq 0)$, and $\rho(s, x)\uparrow$ otherwise. Show that ρ is $\mathscr{K}_1^+$-recursive.

Hint. Take $\varphi = \lambda sx.1$ and $\psi = \psi_2^+ \psi_0(O, I)$. Then $\bar{j}\check{\rho} = \bar{j}\Psi_2^+(\check{\varphi}, \bar{j}\check{\psi})$ for all j, hence $\check{\rho} = D_1 \langle \Psi_2^+ \rangle C_1([I_1]\check{\varphi}, \check{\psi})$.

Remark. It follows that the universal quantification halves ${}^{j+2}E_u$ of all ${}^{j+2}E$ are $\mathscr{K}_1^+$-recursive, where

$$ {}^{j+2}E_u(\alpha^{j+1}) = \begin{cases} \uparrow, & \text{if } \exists \alpha^j(\alpha^{j+1}(\alpha^j) = 0), \\ 1, & \text{if } \forall \alpha^j(\alpha^{j+1}(\alpha^j) \neq 0). \end{cases} $$

CHAPTER 30

Inductive definability theory

Moschovakis [1977] proposes the program of developing Recursion Theory within the framework of Inductive Definability Theory. His approach is well illustrated by the handling of Higher Recursion Theory in Kechris and Moschovakis [1977]. A similar conceptual approach is suggested by Feferman [1977]. We believe the previous chapters of this book support the view that Recursion Theory has its own independent foundations. Moreover, the present chapter shows some notions and results of Inductive Definability Theory to be particular instances of more general ones of Recursion Theory.

We begin with some introductory definitions from Moschovakis [1974].

Assume that an *abstract structure* $\mathscr{V} = (M, \mathscr{R}_1, \ldots, \mathscr{R}_l)$ is given, where M is a set, $\omega \subseteq M$, $\mathscr{R}_i$ is a n_i-ary relation, $\mathscr{R}_i \subseteq M^{n_i}$, $1 \leq i \leq l$. The language $\mathscr{L}^{\mathscr{V}}$ of $\mathscr{V}$ has individual variables $x, y, z, \ldots$, relation constants $\mathscr{R}_1, \ldots, \mathscr{R}_l$ and n-ary relation variables $S, T, U, \ldots$ for all $n \geq 1$. We write $\vec{x}, \vec{S}$ for tuples of individual and relation variables; if $\vec{S} = S_1, \ldots, S_k$, then $\neg\vec{S}$ will stand for $\neg S_1, \ldots, \neg S_k$.

Relation symbols $\mathscr{R}$ are relation constants and variables. The expressions $\mathscr{R}(x_1, \ldots, x_n)$ are formulas and whenever φ, ψ are formulas, then so are $\varphi \,\&\, \psi$, $\varphi \vee \psi, \neg\varphi$, $\exists y\varphi$ and $\forall y\varphi$. We write explicitly $\varphi(\vec{x}, \vec{S})$ to fix a list $\vec{x}, \vec{S}$ which contains all relation and free individual variables in φ. Notice that $\mathscr{L}^{\mathscr{V}}$ is a '*lightface*' language since no individual constants are allowed, while its '*boldface*' version in Moschovakis [1974] has constants for the members of M.

The notion of a relation symbol $\mathscr{R}$ *positive in* φ is introduced as follows. $\mathscr{R}$ is positive in $\mathscr{R}(x_1, \ldots, x_n)$ and all formulas φ which have no occurrences of $\mathscr{R}$; if $\mathscr{R}$ is positive in φ, ψ, then it is positive in $\varphi \,\&\, \psi$, $\varphi \vee \psi$, $\exists y\varphi$ and $\forall y\varphi$. A formula φ is *positive* iff it is free of negation. If $\varphi(\vec{x}, \vec{S})$ is positive, then all the relation variables in $\vec{S}$ are obviously positive in φ.

For all signatures $(n, m_1, \ldots, m_k)$ we have *second order relations*

$$\mathscr{P} = \mathscr{P}(\vec{x}, \vec{S}) \subseteq M^n \times 2^{M^{m_1}} \times \cdots \times 2^{M^{m_k}}$$

First order relations are those with $k = 0$. A relation $\mathscr{P}(\vec{x}, \vec{S})$ is (*positive*) *elementary on* $\mathscr{V}$ iff there is a (positive) formula $\varphi(\vec{x}, \vec{S}) \in \mathscr{L}^{\mathscr{V}}$ such that $\mathscr{P}(\vec{x}, \vec{S})$ iff $\varphi(\vec{x}, \vec{S})$ is true. Given a formula $\varphi(\vec{x}, \vec{S}, S)$ of signature $(n, m_1, \ldots, m_k, n)$ with S positive in φ, a second order relation $\mathscr{I}_\varphi(\vec{x}, \vec{S})$ is constructed by taking

$$\mathscr{I}_\varphi^\xi = \{(\vec{x}, \vec{S}) / \varphi(\vec{x}, \vec{S}, \{\vec{x}' / (\bigcup_{\eta < \xi} \mathscr{I}_\varphi^\eta)(\vec{x}', \vec{S})\})\}$$

and $\mathscr{I}_\varphi = \bigcup_\xi \mathscr{I}_\varphi^\xi$. A relation $\mathscr{P}(\vec{x}, \vec{S})$ is *(positive) inductive on $\mathscr{V}$* iff there is a (positive) formula $\varphi(\vec{y}, \vec{x}, \vec{S}, S)$ and natural numbers $s_1, \ldots, s_m$ such that

$$\mathscr{P}(\vec{x}, \vec{S}) \Leftrightarrow \mathscr{I}_\varphi(s_1, \ldots, s_m, \vec{x}, \vec{S}).$$

It should be stressed that second order relations positive inductive on $\mathscr{V}$ are monotonic in their relation arguments.

While the main goal of this chapter is to establish that inductiveness is a particular instance of IOS-recursiveness, the following statement shows that it is sufficient to deal with the notion of positive inductiveness.

Proposition 30.1. A second order relation $\mathscr{P}(\vec{x}, \vec{S})$ is elementary (inductive) on $\mathscr{V}$ iff there is a relation $Q(\vec{x}, \vec{T}, \vec{S}, \vec{S}')$ positive elementary (respectively, positive inductive) on $\mathscr{V}$ such that

$$\mathscr{P}(\vec{x}, \vec{S}) \Leftrightarrow Q(\vec{x}, \neg \mathscr{R}_1, \ldots, \neg \mathscr{R}_l, \vec{S}, \neg \vec{S}).$$

Proof. This follows from the evident fact that whenever $\varphi(\vec{x}, \vec{S}, \vec{U}) \in \mathscr{L}^{\mathscr{V}}$ and $\vec{U}$ are positive in φ, then there is a positive formula $\psi(\vec{x}, \vec{T}, \vec{S}, \vec{S}', \vec{U})$ such that $\varphi(\vec{x}, \vec{S}, \vec{U})$ is true iff $\psi(\vec{x}, \neg \mathscr{R}_1, \ldots, \neg \mathscr{R}_l, \vec{S}, \neg \vec{S}, \vec{U})$ is true.

Let us design now an IOS suitable for treating second order relations.

Example 30.1. The space $\mathscr{S}_1$ is constructed as follows. Take the set $\mathscr{F}$ of all relations $X \subseteq \omega \times \bigcup_n M^n$ concentrated on M^n for a certain n, where X *concentrates on M^n* iff

$$X(s, x_1, \ldots, x_m) \Leftrightarrow \forall x_{m+1} \ldots \forall x_n X(s, x_1, \ldots, x_n),$$
$$X(s, x_1, \ldots, x_k) \Leftrightarrow X(s, x_1, \ldots, x_n)$$

for all $m \le n \le k$, then take $\mathscr{T} = \omega$,

$$t(X) = \min\{n/X \text{ concentrates on } M^n\},$$

$$X \le Y \text{ iff } X \subseteq Y,$$

$$\Pi(X, Y) = \{(2s, \vec{x})/X(s, \vec{x})\} \cup \{(2s+1, \vec{x})/Y(s, \vec{x})\},$$

$$L'(X) = \{(s, \vec{x})/X(2s, \vec{x})\},\ R'(X) = \{(s, \vec{x})/X(2s+1, \vec{x})\}.$$

Get a $(**)$-complete OS $\mathscr{S}'$ from the $\mathscr{T}$-SCPS $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ by 19.9. Finally, $\mathscr{S}_1 = (\mathscr{F}_1, I_1, \Pi, L_1, R_1)$ is the isomorphic copy of $\mathscr{S}'$ obtained by transforming mappings $\varphi' : \mathscr{F} \to \mathscr{F}$ into relations $\{(s, \vec{x}, Y)/\varphi'(Y)(s, \vec{x})\}$.

We therefore have $\mathscr{F}_1 = \{\phi/\phi \subseteq \omega \times (\bigcup_n M^n) \times \mathscr{F} \mathbin{\&} \phi$ is monotonic and normal$\}$, where ϕ is normal iff there is an n such that whenever Y concentrates on M^m, $m \ge n$, then so does $\{(s, \vec{x})/\phi(s, \vec{x}, Y)\}$,

$$\phi \le \Psi \text{ iff } \phi \subseteq \Psi,$$

$$\phi\Psi(s, \vec{x}, Y) \Leftrightarrow \phi(s, \vec{x}, \lambda t\vec{y}.\Psi(t, \vec{y}, Y)),$$

writing $\lambda t\vec{y}.\Psi(t, \vec{y}, Y)$ for $\{(t, \vec{y})/\Psi(t, \vec{y}, Y)\}$,

$(\phi, \Psi)(s, \vec{x}, Y) \Leftrightarrow s$ is even $\& \ \phi(s/2, \vec{x}, Y) \vee s$ is odd $\&\ \Psi((s-1)/2, \vec{x}, Y)$,

$I_1(s, \vec{x}, Y) \Leftrightarrow Y(s, \vec{x})$, $L_1(s, \vec{x}, Y) \Leftrightarrow Y(2s, \vec{x})$ and $R_1(s, \vec{x}, Y) \Leftrightarrow Y(2s+1, \vec{x})$.

The equality $\bar{n}\langle\phi\rangle = \phi\bar{n}$ implies

$$\langle\phi\rangle(\bar{n}(s), \vec{x}, Y) \Leftrightarrow \phi(s, \vec{x}, \lambda ty.\, Y(\bar{n}(t), \vec{y})),$$

where $\bar{n}(s)$ stands for $2^n(2s+1)-1$ as in example 22.5.

Iteration in higher order spaces is characterized as in 12.25; hence $\lambda s\vec{x}.[\phi](s, \vec{x}, Y)$ is the least X in $\mathscr{F}$ satisfying the equivalence

$$X(s, \vec{x}) \Leftrightarrow s \text{ is even } \& \ Y\left(\frac{s}{2}, \vec{x}\right) \vee s \text{ is odd } \& \ \phi\left(\frac{s-1}{2}\cdot\vec{x}, X\right).$$

Consider the members $\Psi_0, \Psi_\&, \Psi_\vee, \Psi_\exists, \Psi_\forall$ of $\mathscr{F}_1$ such that

$$\begin{aligned}
&\Psi_0(s, \vec{x}, Y) \Leftrightarrow Y(0, \vec{x}),\\
&\Psi_\&(s, \vec{x}, Y) \Leftrightarrow Y(2s, \vec{x}) \,\&\, Y(2s+1, \vec{x}),\\
&\Psi_\vee(s, \vec{x}, Y) \Leftrightarrow Y(2s, \vec{x}) \vee Y(2s+1, \vec{x}),\\
&\Psi_\exists(s, \vec{x}, Y) \Leftrightarrow \exists y\, Y(s, y, \vec{x}),\\
&\Psi_\forall(s, \vec{x}, Y) \Leftrightarrow \forall y\, Y(s, y, \vec{x})
\end{aligned}$$

for all $\vec{x} \in M^n$, $n \geq t(Y)$ in the case of $\Psi_0, \Psi_\&, \Psi_\vee$, respectively $n+1 \geq t(Y)$ in the case of $\Psi_\exists, \Psi_\forall$. (In fact, the equivalences for $\Psi_0, \Psi_\&, \Psi_\vee$ hold for all $\vec{x}$.)

For all $i_0, \ldots, i_m$, $m > 0$, consider the member $\Psi_{i_0,\ldots,i_m}$ of $\mathscr{F}_1$ such that for all Y the relation $\lambda s\vec{x}.\Psi_{i_0,\ldots,i_m}(s, \vec{x}, Y)$ concentrates on M^n, $n = \max\{i_0, \ldots, i_m\}$, and

$$\Psi_{i_0,\ldots,i_m}(x_0, \ldots, x_n, Y) \Leftrightarrow Y(x_{i_0}, \ldots, x_{i_m}).$$

Fix the set of initial elements

$$\mathscr{M}_1 = \{\Psi_0, \Psi_\&, \Psi_\vee, \Psi_\exists, \Psi_\forall, \Psi_{i_0,\ldots,i_m} / m > 0 \,\&\, i_0, \ldots, i_m \in \omega\}.$$

One last definition completes our preliminaries. To each second order relation $\mathscr{P}(\vec{x}, \vec{S})$ of signature $(n, m_1, \ldots, m_k)$ which is monotonic on $\vec{S}$ assign the unique $\mathscr{P}^* \in \mathscr{F}_1$ such that for all Y the relation $\lambda s\vec{x}.\mathscr{P}^*(s, \vec{x}, Y)$ concentrates on M^n and

$$\mathscr{P}^*(s, \vec{x}, Y) \Leftrightarrow \mathscr{P}(\vec{x}, \lambda_{m_1}\vec{y}.\bar{0}(0, \vec{y}, Y), \ldots, \lambda_{m_{k-1}}\vec{y}.\overline{k-2}(0, \vec{y}, Y), \lambda_{m_k}\vec{y}.R_1^{k-1}(0, \vec{y}, Y)$$

for all s, Y and $\vec{x} \in M^n$, writing $\lambda_m\vec{y}\cdot$——for $\{\vec{y} \in M^m/$——$\}$. In particular, $\mathscr{R}^*(s, \vec{x}, Y) \Leftrightarrow \mathscr{R}(\vec{x})$ for all first order relations $\mathscr{R}$. Notice that

$$\begin{gathered}
\mathscr{P}^*(\phi_1, \ldots, \phi_k)(s, \vec{x}, Y) \Leftrightarrow \mathscr{P}(\vec{x}, \lambda_{m_1}\vec{y}\cdot\phi_1(0, \vec{y}, Y), \ldots, \lambda_{m_k}\vec{y}\cdot\phi_k(0, \vec{y}, Y)),\\
\mathscr{P}(\vec{x}, \vec{S}) \Leftrightarrow \mathscr{P}^*(S_1^*, \ldots, S_k^*)(s, \vec{x}, Y)
\end{gathered}$$

for all s, Y and $\vec{x} \in M^n$.

Proposition 30.2. If $\mathscr{P}(\vec{x}, \vec{S})$ is positive elementary on $\mathscr{V}$, then $\mathscr{P}^*$ is polynomial in $\{\mathscr{R}_1^*, \ldots, \mathscr{R}_l^*\} \cup \mathscr{M}_1$.

Proof. We follow the construction of the positive formula φ which defines $\mathscr{P}(x_1, \ldots, x_n, S_1, \ldots, S_k)$.

1. Let $\varphi(\vec{x}, \vec{S})$ be $S_i(x_{i_1}, \ldots, x_{i_m})$, where $1 \leq i \leq k$, $1 \leq i_1, \ldots, i_m \leq n$. Then $\mathscr{P}^* = \Psi_0\overline{i-1}\Psi_{0,i_1,\ldots,i_m}$, if $i < k$, and $\mathscr{P}^* = \Psi_0 R_1^{k-1}\Psi_{0,i_1,\ldots,i_m}$, if $i = k$.

2. Let $\varphi(\vec{x}, \vec{S})$ be $\mathscr{R}_i(x_{i_1}, \ldots, x_{i_m})$, where $1 \le i \le l$, $1 \le i_1, \ldots, i_m \le n$. Then $\mathscr{P}^* = \Psi_{0,i_1,\ldots,i_m}\mathscr{R}_i^*$.

3. Let $\mathscr{P}_1(\vec{x}, \vec{S})$, $\mathscr{P}_2(\vec{x}, \vec{S})$ be defined respectively by φ, ψ. If $\mathscr{P}$ is defined by $\varphi \& \psi$, then

$$\mathscr{P}^*(s, \vec{x}, Y) \Leftrightarrow \mathscr{P}_1^*(s, \vec{x}, Y) \& \mathscr{P}_2^*(s, \vec{x}, Y)$$

for all s, Y and $\vec{x} \in M^n$; hence $\mathscr{P}^* = \Psi_{\&}(\mathscr{P}_1^*, \mathscr{P}_2^*)$. Similarly, if $\mathscr{P}$ is defined by $\varphi \vee \psi$, $\exists y \varphi$ or $\forall y \varphi$, then $\mathscr{P}^* = \Psi_{\vee}(\mathscr{P}_1^*, \mathscr{P}_2^*)$, $\mathscr{P}^* = \Psi_{\exists}\mathscr{P}_1^*$, $\mathscr{P}^* = \Psi_{\forall}\mathscr{P}_1^*$ respectively. The proof is complete.

Proposition 30.3. If $\mathscr{P}(\vec{x}, \vec{S})$ is positive inductive on $\mathscr{V}$, then $\mathscr{P}^*$ is recursive in $\{\mathscr{R}_1^*, \ldots, \mathscr{R}_l^*\} \cup \mathscr{M}_1$.

Proof. There is a positive formula $\varphi(\vec{y}, \vec{x}, \vec{S}, S)$ and numbers $s_1, \ldots, s_m$ such that

$$\mathscr{P}(\vec{x}, \vec{S}) \Leftrightarrow \mathscr{I}_\varphi(s_1, \ldots, s_m, \vec{x}, \vec{S}).$$

Let $\mathscr{Q}(\vec{y}, \vec{x}, \vec{S}, S)$ be the positive elementary on $\mathscr{V}$ relation of signature $(m+n, m_1, \ldots, m_k, m+n)$ defined by φ. Write $(\vec{z}, Y)^\nabla$ for

$$\vec{z}, \lambda_{m_1}\vec{z}'.\bar{0}(0, \vec{z}'\, Y), \ldots, \lambda_{m_{k-1}}\vec{z}'.\overline{k-2}\,(0, \vec{z}', Y), \lambda_{m_k}\vec{z}'.R_1^{k-1}(0, \vec{z}', Y),$$

$\vec{z}$ ranging over M^{m+n}. Then

$$\begin{aligned} \mathscr{I}_\varphi^*(s, \vec{y}, \vec{x}, Y) &\Leftrightarrow \mathscr{I}_\varphi((\vec{y}, \vec{x}, Y)^\nabla) \Leftrightarrow \mathscr{Q}((\vec{y}, \vec{x}, Y)^\nabla, \lambda_{m+n}\vec{z}.\mathscr{I}_\varphi((\vec{z}, Y)^\nabla)) \\ &\Leftrightarrow \mathscr{Q}((\vec{y}, \vec{x}, Y)^\nabla, \lambda_{m+n}\vec{z}.\mathscr{I}_\varphi^*(0, \vec{z}, Y)) \\ &\Leftrightarrow \mathscr{Q}^*(\bar{0}, \ldots, \overline{k-2}, R_1^{k-1}, \mathscr{I}_\varphi^*)(s, \vec{y}, \vec{x}, Y) \end{aligned}$$

for all s, Y and $\vec{y} \in M^m, \vec{x} \in M^n$. Both

$$\lambda s\vec{z}.\mathscr{I}_\varphi^*(s, \vec{z}, Y),\ \lambda s\vec{z}.\mathscr{Q}^*(\bar{0}, \ldots, \overline{k-2}, R_1^{k-1}, \mathscr{I}_\varphi^*)(s, \vec{z}, Y)$$

concentrate on M^{m+n} for all Y, hence

$$\mathscr{I}_\varphi^* = \mathscr{Q}^*(\bar{0}, \ldots, \overline{k-2}, R_1^{k-1}, \mathscr{I}_\varphi^*).$$

Suppose that $\Theta = \mathscr{Q}^*(\bar{0}, \ldots, \overline{k-2}, R_1^{k-1}, \Theta)$. The relation $\lambda s\vec{z}.\mathscr{Q}^*(\bar{0}, \ldots, \overline{k-2}, R_1^{k-1}, \Theta)(s, \vec{z}, Y)$ concentrates on M^{m+n} for all Y, hence so does $\lambda s\vec{z}.\Theta(s, \vec{z}, Y)$. It follows that

$$\begin{aligned} \Theta(s, \vec{y}, \vec{x}, Y) &\Leftrightarrow \mathscr{Q}^*(\bar{0}, \ldots, \overline{k-2}, R_1^{k-1}, \Theta)(s, \vec{y}, \vec{x}, Y) \\ &\Leftrightarrow \mathscr{Q}((\vec{y}, \vec{x}, Y)^\nabla, \lambda_{m+n}\vec{z}.\Theta(0, \vec{z}, Y)), \end{aligned}$$

hence $\Theta(s, \vec{y}, \vec{x}, Y) \Leftrightarrow \Theta(0, \vec{y}, \vec{x}, Y)$ and $\mathscr{I}_\varphi((\vec{y}, \vec{x}, Y)^\nabla) \Rightarrow \Theta(s, \vec{y}, \vec{x}, Y)$, i.e. $\mathscr{I}_\varphi^*(s, \vec{y}, \vec{x}, Y) \Rightarrow \Theta(s, \vec{y}, \vec{x}, Y)$ for all s, Y and $\vec{y} \in M^m$, $\vec{x} \in M^n$. Observing that both $\lambda s\vec{z}.\mathscr{I}_\varphi^*(s, \vec{z}, Y)$ and $\lambda s\vec{z}.\Theta(s, \vec{z}, Y)$ concentrate on M^{m+n}, we conclude that $\mathscr{I}_\varphi^* \le \Theta$. Therefore,

$$\mathscr{I}_\varphi^* = \mu\Theta.\mathscr{Q}^*(\bar{0}, \ldots, \overline{k-2}, R_1^{k-1}, \Theta),$$

hence $\mathscr{I}_\varphi^*$ is recursive in $\{\mathscr{R}_1^*, \ldots, \mathscr{R}_l^*\} \cup \mathscr{M}_1$ by 6.11 since so is $\mathscr{Q}^*$ by 30.2.

The proof of 22.3 implies $\Delta(R_1, L_1)(s, \vec{x}, Y) \Leftrightarrow Y(s+1, \vec{x})$. Multiplying $\Psi_0, \Delta(R_1, L_1)$ and $\Psi_{i_0,\dots,i_u}$ for appropriate $u, i_0, \dots, i_u$, one gets a ϕ such that $\lambda s\vec{x}.\phi(s, \vec{x}, Y)$ concentrates on M^n and

$$\phi(s, x_1, \dots, x_n, Y) \Leftrightarrow Y(s, s_1, \dots, s_m, x_1, \dots, x_n)$$

for all Y. Therefore, $\mathscr{P}^* = \phi.\mathscr{I}^*_\varphi$; hence $\mathscr{P}^*$ is recursive in $\{\mathscr{R}^*_1, \dots, \mathscr{R}^*_l\} \cup \mathscr{M}_1$, which completes the proof.

The structure $\mathscr{V}$ is said to be *ω-acceptable* iff the subset ω of M admits a *coding scheme* $\mathscr{C} = (\omega, \leq, \langle\ \ \rangle)$ such that the relations ω, $\leq$, Seq, lh, q and their negations are positive elementary on $\mathscr{V}$, where $\leq$ is the ordinary total ordering of ω, $\langle\ \ \rangle : \bigcup_n \omega^n \to \omega$ is injective,

$$\begin{aligned} \mathrm{Seq}(x) &\Leftrightarrow \exists n \exists x_1 \dots \exists x_n (x = \langle x_1, \dots, x_n \rangle), \\ lh(x, y) &\Leftrightarrow y \in \omega \,\&\, x = \langle x_1, \dots, x_y \rangle, \\ q(x, y, z) &\Leftrightarrow y \in \omega \,\&\, x = \langle x_1, \dots, x_n \rangle \,\&\, z = x_y. \end{aligned}$$

By an argument of Moschovakis [1974], this ensures that all the arithmetical relations over ω are positive elementary on $\mathscr{V}$. In particular, this is the case for the following recursive ones:

$$\begin{aligned} F_1(x, y) &\Leftrightarrow x \in \omega \,\&\, y = 2x, \\ F_2(x, y) &\Leftrightarrow x \in \omega \,\&\, y = 2x + 1, \\ F_3(x, y, z) &\Leftrightarrow \exists u \exists v (x + 1 = 2^u(2v+1) \,\&\, z + 1 = 2^u(2y+1)). \end{aligned}$$

Proposition 30.4. Let $\mathscr{V}$ be ω-acceptable and ϕ be primitive in $\{\mathscr{R}^*_1, \dots, \mathscr{R}^*_l\} \cup \mathscr{M}_1$. Then for all n the relation $\mathscr{P}_{\phi,n}$ of signature $(n+1, n+1)$ is positive elementary on $\mathscr{V}$, where

$$\mathscr{P}_{\phi,n}(\vec{x}, S) \Leftrightarrow \phi(\vec{x}, \tilde{S}),$$

$\tilde{S}$ standing for the unique $X \in \mathscr{F}$ to concentrate on M^n such that $X \cap \omega \times M^n = S \cap \omega \times M^n$.

Proof. By induction on the construction of ϕ.

1. Let $\phi = L_1$. Then

$$\begin{aligned} \mathscr{P}_{\phi,n}(s, x_1, \dots, x_n, S) &\Leftrightarrow L_1(s, x_1, \dots, x_n, \tilde{S}) \Leftrightarrow \tilde{S}(2s, x_1, \dots, x_n) \\ &\Leftrightarrow S(2s, x_1, \dots, x_n) \Leftrightarrow \exists x (F_1(s, x) \,\&\, S(x, x_1, \dots, x_n)); \end{aligned}$$

hence $\mathscr{P}_{\phi,n}$ is positive elementary on $\mathscr{V}$.

2. Let $\phi = \mathscr{R}^*_i$. Taking a $k \geq n_i, n$, one gets

$$\mathscr{P}_{\phi,n}(s, x_1, \dots, x_n, S) \Leftrightarrow \mathscr{R}^*_i(s, x_1, \dots, x_n, \tilde{S}) \Leftrightarrow \forall x_{n+1} \dots \forall x_k \mathscr{R}_i(x_1, \dots, x_{n_i});$$

hence $\mathscr{P}_{\phi,n}$ is positive elementary on $\mathscr{V}$.

3. Let $\phi = \Psi_\exists$. Then $\mathscr{P}_{\phi,0}(s, S) \Leftrightarrow S(s)$ and

$$\begin{aligned} \mathscr{P}_{\phi,n}(s, x_1, \dots, x_n, S) &\Leftrightarrow \Psi_\exists(s, x_1, \dots, x_n, \tilde{S}) \Leftrightarrow \exists y \tilde{S}(s, y, x_1, \dots, x_n) \\ &\Leftrightarrow \exists y S(s, y, x_1, \dots, x_{n-1}) \text{ for } n > 0. \end{aligned}$$

4. The cases $R_1, \Psi_0, \Psi_\&, \Psi_\vee, \Psi_\forall, \Psi_{i_0,\dots,i_m}$ are treated similarly.
5. Let $\mathscr{P}_{\phi,n}$, $\mathscr{P}_{\Psi,n}$ be positive elementary on $\mathscr{V}$ for all n. Take a $k \geq n$ such

that whenever Y concentrates on M^k, then so do $\lambda t\vec{y}.\Psi(t,\vec{y},Y)$ and $\lambda s\vec{x}.\phi(s,\vec{x},Y)$; hence $\lambda t\vec{y}.\Psi(t,\vec{y},Y)=(\lambda_{k+1}t\vec{y}.\Psi(t,\vec{y},Y))^{\sim}$. It follows that

$$\begin{aligned}
&\mathscr{P}_{\phi\Psi,n}(s,x_1,\dots,x_n,S)\Leftrightarrow\phi\Psi(s,x_1,\dots,x_n,\tilde{S})\\
&\quad\Leftrightarrow\phi(s,x_1,\dots,x_n,\lambda t\vec{y}.\Psi(t,\vec{y},\tilde{S}))\\
&\quad\Leftrightarrow\phi(s,x_1,\dots,x_n,(\lambda_{k+1}t\vec{y}.\Psi(t,\vec{y},\tilde{S}))^{\sim})\\
&\quad\Leftrightarrow\phi(s,x_1,\dots,x_n,(\lambda_{k+1}t\vec{y}.\mathscr{P}_{\Psi,k}(t,\vec{y},S\times M^{k-n}))^{\sim})\\
&\quad\Leftrightarrow\forall x_{n+1}\dots\forall x_k\phi(s,x_1,\dots,x_k,(\lambda_{k+1}t\vec{y}.\mathscr{P}_{\Psi,k}(t,\vec{y},S\times M^{k-n}))^{\sim})\\
&\quad\Leftrightarrow\forall x_{n+1}\dots\forall x_k\mathscr{P}_{\phi,k}(s,x_1,\dots,x_k,\lambda_{k+1}t\vec{y}.\mathscr{P}_{\Psi,k}(t,\vec{y},S\times M^{k-n})),
\end{aligned}$$

hence $\mathscr{P}_{\phi\Psi,n}$ is positive elementary on $\mathscr{V}$.

6. Let $\mathscr{P}_{\phi,n},\mathscr{P}_{\Psi,n}$ be positive elementary on $\mathscr{V}$. Then

$$\begin{aligned}
&\mathscr{P}_{(\phi,\Psi),n}(s,x_1,\dots,x_n,S)\\
&\quad\Leftrightarrow(\phi,\Psi)(s,x_1,\dots,x_n,\tilde{S})\\
&\quad\Leftrightarrow\exists x(F_1(x,s)\&\phi(x,x_1,\dots,x_n,\tilde{S})\vee F_2(x,s)\&\Psi(x,x_1,\dots,x_n,\tilde{S}))\\
&\quad\Leftrightarrow\exists x(F_1(x,s)\&\mathscr{P}_{\phi,n}(x,x_1,\dots,x_n,S)\vee F_2(x,s)\&\mathscr{P}_{\Psi,n}(x,x_1,\dots,x_n,S)),
\end{aligned}$$

hence $\mathscr{P}_{(\phi,\Psi),n}$ is positive elementary on $\mathscr{V}$.

7. Let $\mathscr{P}_{\phi,n}$ be positive elementary on $\mathscr{V}$. Then

$$\begin{aligned}
&\mathscr{P}_{\langle\phi\rangle,n}(s,x_1,\dots,x_n,S)\\
&\quad\Leftrightarrow\langle\phi\rangle(s,x_1,\dots,x_n,\tilde{S})\\
&\quad\Leftrightarrow\exists u\exists v(F_3(s,u,s)\&\phi(u,x_1,\dots,x_n,\lambda t\vec{y}.F_3(s,t,v)\&\tilde{S}(v,\vec{y})))\\
&\quad\Leftrightarrow\exists u\exists v(F_3(s,u,s)\&\phi(u,x_1,\dots,x_n,(\lambda_{n+1}t\vec{y}.F_3(s,t,v)\&S(v,\vec{y}))^{\sim}))\\
&\quad\Leftrightarrow\exists u\exists v(F_3(s,u,s)\&\mathscr{P}_{\phi,n}(u,x_1,\dots,x_n,\lambda_{n+1}t\vec{y}.F_3(s,t,v)\&S(v,\vec{y}))),
\end{aligned}$$

hence $\mathscr{P}_{\langle\phi\rangle,n}$ is positive elementary on $\mathscr{V}$. The proof is complete.

Proposition 30.5. Let $\mathscr{V}$ be ω-acceptable and ϕ recursive in $\{R_1^*,\dots,\mathscr{R}_l^*\}\cup\mathscr{M}_1$. Then $\mathscr{P}_{\phi,n}$ is positive inductive on $\mathscr{V}$ for all n.

Proof. The Normal form Theorem 9.3 gives $\phi=\mathrm{I}[\Psi]$ with a certain Ψ primitive in $\{\mathscr{R}_1^*,\dots,\mathscr{R}_l^*\}\cup\mathscr{M}_1$. It suffices to show that $\mathscr{P}_{\phi,n}$ is positive inductive on $\mathscr{V}$ for all n not less than the type of Ψ. Indeed, the type of ϕ is not greater than such n; hence

$$\mathscr{P}_{\phi,m}(s,x_1,\dots,x_m,S)\Leftrightarrow\forall x_{m+1}\dots\forall x_n\mathscr{P}_{\phi,n}(s,x_1,\dots,x_n,S\times M^{n-m})$$

for all $m\le n$.

Take $\Sigma=[\Psi]$. If Y concentrates on M^n, then so do $\lambda s\vec{x}.\Psi(s,\vec{x},Y)$, $\lambda t\vec{y}.\Sigma(t,\vec{y},Y)$ since the types of Ψ,Σ are not greater than n. Using the equality $\Sigma=(I_1,\Psi\Sigma)$ and the fact that $\mathscr{P}_{\Psi,n}$ is positive elementary on $\mathscr{V}$ by 30.4, one gets

$$\begin{aligned}
&\mathscr{P}_{\Sigma,n}(s,x_1,\dots,x_n,S)\Leftrightarrow\Sigma(s,x_1,\dots,x_n,\tilde{S})\\
&\quad\Leftrightarrow\exists x(F_1(x,s)\&\tilde{S}(x,x_1,\dots,x_n)\vee F_2(x,s)\\
&\qquad\&\Psi(x,x_1,\dots,x_n,\lambda t\vec{y}.\Sigma(t,\vec{y},\tilde{S})))
\end{aligned}$$

$$\begin{aligned}&\Leftrightarrow \exists x(F_1(x,s) \& S(x,x_1,\ldots,x_n) \vee F_2(x,s)\\ &\quad \& \Psi(x,x_1,\ldots,x_n,(\lambda_{n+1} t\overline{y}.\mathscr{P}_{\Sigma,n}(t,\overline{y},S))^{\sim}))\\ &\Leftrightarrow \exists x(F_1(x,s) \& S(x,x_1,\ldots,x_n) \vee F_2(x,s)\\ &\quad \& \mathscr{P}_{\Psi,n}(x,x_1,\ldots,x_n,\lambda_{n+1} t\overline{y}.\mathscr{P}_{\Sigma,n}(t,\overline{y},S)))\\ &\Leftrightarrow \mathscr{Q}(s,x_1,\ldots,x_n,S,\lambda_{n+1} t\overline{y}.\mathscr{P}_{\Sigma,n}(t,\overline{y},S))\end{aligned}$$

with a certain relation $\mathscr{Q}$ positive elementary on $\mathscr{V}$.

Suppose that

(1) $\mathscr{P}(s,x_1,\ldots,x_n,S) \Leftrightarrow \mathscr{Q}(s,x_1,\ldots,x_n,S,\lambda_{n+1} t\overline{y}.\mathscr{P}(t,\overline{y},S))$.

Take the unique $\phi_1,\phi_2 \in \mathscr{F}_1$ such that for all Y the relations $\lambda s\overline{x}.\phi_1(s,\overline{x},Y)$, $\lambda s\overline{x}.\phi_2(s,\overline{x},Y)$ concentrate on M^n and

$$\begin{aligned}&\phi_1(s,x_1,\ldots,x_n,Y) \Leftrightarrow Y(s,x_1,\ldots,x_n),\\ &\phi_2(s,x_1,\ldots,x_n,Y) \Leftrightarrow \mathscr{P}(s,x_1,\ldots,x_n,Y \cap \omega \times M^n).\end{aligned}$$

Then $\phi_2 = (\phi_1, \Psi\phi_2)$, hence $\Sigma\phi_1 \le \phi_2$ by (££), which implies $\mathscr{P}_{\Sigma,n}(s,x_1,\ldots,x_n,S) \Rightarrow \mathscr{P}(s,x_1,\ldots,x_n,S)$ for all $s,x_1,\ldots,x_n,S$. Therefore, $\mathscr{P}_{\Sigma,n}$ is the least relation satisfying (1); hence it is positive inductive on $\mathscr{V}$. Finally, $\mathscr{P}_{\phi,n}$ is positive inductive on $\mathscr{V}$ since

$$\mathscr{P}_{\phi,n}(s,x_1,\ldots,x_n,S) \Leftrightarrow \exists y \exists z(F_1(s,y) \& F_2(y,z) \& \mathscr{P}_{\Sigma,n}(z,x_1,\ldots,x_n,S)).$$

The proof is complete.

The following Positive Inductiveness Theorem establishes the desired characterization of positive inductiveness.

Proposition 30.6. Let $\mathscr{V}$ be ω-acceptable. Then a second order relation $\mathscr{P}(\overline{x},\overline{S})$ is positive inductive on $\mathscr{V}$ iff $\mathscr{P}^*$ is recursive in $\{\mathscr{R}_1^*,\ldots,\mathscr{R}_l^*\} \cup \mathscr{M}_1$.

Proof. The 'only if'-part of the equivalence follows by 30.3.

Assume that $\mathscr{P}$ is of signature $(n,m_1,\ldots,m_k)$, monotonic in its relation arguments, and $\mathscr{P}^*$ is recursive in $\{\mathscr{R}_1^*,\ldots,\mathscr{R}_l^*\} \cup \mathscr{M}_1$. Fix a number $m \ge n, m_1,\ldots,m_k$. Then

$$\begin{aligned}&\mathscr{P}(x_1,\ldots,x_n,S_1,\ldots,S_k)\\ &\quad \Leftrightarrow \mathscr{P}^*(0,x_1,\ldots,x_n,\lambda t\overline{y}.(S_1^*,\ldots,S_k^*)(t,\overline{y},O))\\ &\quad \Leftrightarrow \mathscr{P}^*(0,x_1,\ldots,x_n,(\lambda_{m+1} t\overline{y}.(S_1^*,\ldots,S_k^*)(t,\overline{y},O))^{\sim})\\ &\quad \Leftrightarrow \forall x_{n+1}\ldots\forall x_m \mathscr{P}_{\mathscr{P}^*,m}(0,x_1,\ldots,x_m,\lambda_{m+1} t\overline{y}.(S_1^*,\ldots,S_k^*)(t,\overline{y},O))\\ &\quad \Leftrightarrow \forall x_{n+1}\ldots\forall x_m \mathscr{P}_{\mathscr{P}^*,m}(0,x_1,\ldots,x_m,\lambda_{m+1} t\overline{y}.\mathscr{Q}(t,\overline{y},S_1,\ldots,S_k)),\end{aligned}$$

where

$$\begin{aligned}&\mathscr{Q}(t,y_1,\ldots,y_m,S_1,\ldots,S_k)\\ &\quad \Leftrightarrow \exists y(t+1 = 2^0(2y+1) \& S_1(y_1,\ldots,y_{m_1}) \vee \cdots \vee t+1 = 2^{k-2}(2y+1)\\ &\qquad \& S_{k-1}(y_1,\ldots,y_{m_{k-1}}) \vee t+1 = 2^{k-1}(y+1) \& S_k(y_1,\ldots,y_{m_k})).\end{aligned}$$

Therefore, $\mathscr{P}$ is positive inductive on $\mathscr{V}$ since $\mathscr{P}_{\mathscr{P}^*,m}$ is by 30.5 and positive inductiveness is preserved by substitutions and negation-free explicit definitions. The proof is complete.

Proposition 30.6 makes it possible to transfer results of the general IOS-theory to the theory of inductive relations: for example the Enumeration Theorem 9.18 and the First Recursion Theorem 9.13* for $\mathscr{S}_1$. The latter is a nontrivial generalization of the Positive Induction Completeness Theorem 6B.4 of Moschovakis [1974] (cf. the first Recursion Theorem in example 28.2).

The space $\mathscr{S}_1$ is equally suitable for treating more general notions of inductiveness introduced e.g. by adding a monotonic quantifier Q to the language $\mathscr{L}^{\mathscr{V}}$. One just adds corresponding elements Ψ_Q, Ψ_Q to $\mathscr{M}_1$.

It is worth mentioning finally that the abstract structures $\mathscr{V}$ considered in this chapter are not necessarily acceptable in the sense of Moschovakis [1974], though acceptability implies ω-acceptability and helps to simplify some formulations and proofs.

EXERCISES TO CHAPTER 30

Exercise 30.1. Show that the elements $\Psi_\vee, \Psi_\&$ of example 30.1 satisfy respectively conditions (1) of exercises 7.10, 7.14.

Example 30.2. Take a set $M, \omega \subseteq M$, fix an injective function $J: M^2 \to M$, then take $\mathscr{F} = \{X/X \subseteq M\}$, $X \leq Y$ iff $X \subseteq Y$, $\Pi(X,Y) = J(0,X) \cup J(1,Y)$, $L'(X) = \{y/X(\langle 0,y\rangle)\}$ and $R'(X) = \{y/X(\langle 1,y\rangle)\}$, writing $\langle x,y\rangle$ for $J(x,y)$. Construct a $(**)_0$-complete OS $\mathscr{S}'$ from the SCPS $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ by 19.9. Finally, $\mathscr{S}_1$ is the isomorphic copy of $\mathscr{S}'$ obtained by transforming mappings $\varphi': \mathscr{F} \to \mathscr{F}$ into second order relations $\{(x,Y)/\varphi'(Y)(x)\}$. Therefore, $\mathscr{F}_1 = \{\phi/\phi \subseteq M \times \mathscr{F} \,\&\, \phi \text{ is monotonic}\}$,

$$\phi\Psi(x,Y) \Leftrightarrow \phi(x, \lambda y.\Psi(y,Y)),$$
$$(\phi,\Psi)(x,Y) \Leftrightarrow \exists y(x = \langle 0,y\rangle \,\&\, \phi(y,Y) \vee x = \langle 1,y\rangle \,\&\, \Psi(y,Y)),$$
$$I_1(x,Y) \Leftrightarrow Y(x), L_1(x,Y) \Leftrightarrow Y(\langle 0,x\rangle) \text{ and } R_1(x,Y) \Leftrightarrow Y(\langle 1,x\rangle).$$

Exercise 30.2. Let $\mathscr{S}_1$ be the IOS of example 30.2 and let be $St: \mathscr{F}_1 \to \mathscr{F}_1$ such that

$$St(\phi)(x,Y) \Leftrightarrow \exists yz(x = \langle y,z\rangle \,\&\, \phi(z, \lambda u.\, Y(\langle y,u\rangle))).$$

Show that St is a t-operation satisfying the axiom $\mathrm{t}\mu\mathrm{A}_3$.

Hint. Take $\mathscr{L} = M^\sim$, where $\tilde{x}(y,Y) \Leftrightarrow Y(\langle x,y\rangle)$,

$$K_0(x,Y) \Leftrightarrow \exists yz(x = \langle y, \langle 0,z\rangle\rangle \,\&\, Y(\langle 0, \langle y,z\rangle\rangle)$$
$$\vee\, x = \langle y, \langle 1,z\rangle\rangle \,\&\, Y(\langle 1, \langle y,z\rangle\rangle)).$$
$$K_1(x,Y) \Leftrightarrow \exists yzu(x = \langle y, \langle z,u\rangle\rangle \,\&\, Y(\langle\langle y,z\rangle, u\rangle)),$$
$$K_2(x,Y) \Leftrightarrow \exists yzu(x = \langle\langle y,z\rangle, u\rangle \,\&\, Y(\langle y, \langle z,u\rangle\rangle)).$$

Use 10.18, 18.21.

Exercise 30.3. Show that the above operation St satisfies the assumptions of exercise 10.9 for appropriate $K_3 - K_6 \in \mathscr{F}_1$ expressible by $L_1, R_1, K_0^* - K_2^*$, multiplication, Π_1 and St, where

$$K_0^*(x, Y) \Leftrightarrow Y(\langle x, x \rangle),$$
$$K_1^*(x, Y) \Leftrightarrow \exists yz(x = \langle y, z \rangle \& Y(\langle z, y \rangle)),$$
$$K_2^*(x, Y) \Leftrightarrow \exists yz(x = \langle y, z \rangle \& Y(z)).$$

Hint. Cf. example 24.1.

Since $K_0 - K_2$ are also expressible by $K_0^* - K_2^*$, let us extend the notion of st-recursiveness so that ϕ is st-recursive in $\mathscr{B}_1 \subseteq \mathscr{F}_1$ iff

$$\phi \in cl(\{L_1, R_1, K_0^* - K_2^*\} \cup \mathscr{B}_1/{\circ}, \Pi_1, \langle \quad \rangle, [\quad], St).$$

st-Recursiveness provides a simpler characterization of the inductive relations over acceptable structures, where $\mathscr{V}$ is *acceptable* iff it admits a *coding scheme* $\mathscr{C} = (\omega, \leq, \langle \quad \rangle), \langle \quad \rangle : \bigcup_n M^n \to M$, with relations ω, $\leq$, Seq, lh, q and their negations positive elementary on $\mathscr{V}$.

Take $\mathscr{M}_1 = \{\Psi_s, \Psi_\&, \Psi_\vee, \Psi_\exists, \Psi_\forall\}$, where

$$\Psi_s(x, Y) \Leftrightarrow \exists yz(x = \langle y, z \rangle \& y \in \omega \& Y(\langle y+1, z \rangle)),$$
$$\Psi_\& = L_1 \cap R_1, \quad \Psi_\vee = L_1 \cup R_1,$$
$$\Psi_\exists(x, Y) \Leftrightarrow \exists y Y(\langle y, x \rangle), \quad \Psi_\forall(x, Y) \Leftrightarrow \forall y Y(\langle y, x \rangle).$$

To each relation $\mathscr{P}$ of signature $(n, m_1, \ldots, m_k)$ assign a $\mathscr{P}^\sim \subseteq M \times \mathscr{F}$ such that

$$\mathscr{P}^\sim(x, Y) \Leftrightarrow \exists x_1 \ldots x_n(x = \langle x_1, \ldots, \langle x_{n-1}, x_n \rangle \cdots \rangle \& \mathscr{P}(x_1, \ldots, x_n, S_1, \ldots, S_k)).$$

where $S_i(z_1, \ldots, z_{m_i}) \Leftrightarrow Y(\langle i, \langle z_1, \ldots, \langle z_{m_i-1}, z_{m_i} \rangle \cdots \rangle\rangle)$. In particular, if $k = 0$, then

$$\mathscr{P}^\sim(x, Y) \Leftrightarrow \exists x_1 \ldots x_n(x = \langle x_1, \ldots, \langle x_{n-1}, x_n \rangle \cdots \rangle \& \mathscr{P}(x_1, \ldots, x_n)).$$

Exercise 30.4. Let $\mathscr{V} = (M, \mathscr{R}_1, \ldots, \mathscr{R}_l)$ be acceptable. Fix a coding scheme $\mathscr{C}$ positive elementary on $\mathscr{V}$, then take the space $\mathscr{S}_1$ of example 30.2 based on M and $J = \lambda xy. \langle x, y \rangle$. Show that a second order relation $\mathscr{P}$ is positive inductive on $\mathscr{V}$ iff $\mathscr{P}^\sim$ is st-recursive in $\{\mathscr{R}_1^\sim, \ldots, \mathscr{R}_l^\sim\} \cup \mathscr{M}_1$. In particular, $\mathscr{P} \subseteq M \times 2^M$ is positive inductive on $\mathscr{V}$ iff it is st-recursive in $\{\mathscr{R}_1^\sim, \ldots, \mathscr{R}_l^\sim\} \cup \mathscr{M}_1$.

Remark. To incorporate the original 'boldface' inductiveness of Moschovakis [1974] one should therefore substitute $\mathscr{L} \cup \mathscr{M}_1$ for $\mathscr{M}_1$ and turn to the 'boldface' version of IOS-theory referred to in the remarks to exercise 10.9.

PART G

Epilogue

Following Kreisel [1971], the main purposes of axiomatizing (fragments of) Recursion Theory are: (i) To help advance and better understand other parts of logic and mathematics, especially Ordinary Recursion Theory and its specific generalizations known as Generalized Recursion Theory, as well as Theoretical Computer Science; (ii) To analyse the concept of effective computability; (iii) Ultimately, to provide both independent foundations and genuine axiomatic development of Recursion Theory.

Axiomatic approaches to Recursion Theory have been suggested or studied by Wagner [1969], Strong [1968], Friedman [1971a], Moschovakis [1971], Moldestad [1977], Fenstad [1980], Skordev [1980], Ivanov [1980], Fitting [1981], Zashev [1983] and others. Especially elaborate and deep is the exploration of the so called computation theories in Fenstad's book. Our discussion will concentrate on the present approach and those of Skordev and Zashev which form a separate trend, Algebraic Recursion Theory (ART for short).

Although it would be overoptimistic to expect early contributions of ART to other parts of logic and mathematics, some encouraging examples can be pointed out.

Let $\mathscr{F}$ be a class of elements together with a notion of effective computability, the subclass $\mathscr{U}$ consisting of all the computable elements. Informally speaking, if there exist elements $\mathscr{B}_0$ and finitely many operations $\mathscr{B}'$ simple enough to be assumed initial and satisfying

$$\mathscr{U} = cl(\mathscr{B}_0/\mathscr{B}'), \tag{1}$$

then the notion of effective computability concerned is *structurizable*. For instance, the notion of μ-recursiveness is structurized by Kleene's definition adduced in chapter 2. (To be more precise, by its modification for unary functions given in the same chapter.) As an application of ART one obtains from 24.3 or by the corresponding result of Skordev [1980] that Moschovakis' prime computability is structurizable, a by no means obvious fact. Normal forms for prime and search computable functions are also obtained.

Theorem 29.4 however does not imply that the Kleene-recursiveness in higher types is structurizable which would indeed be surprising if true. One can easily get (1) by taking in $\mathscr{B}_0$ the universal function σ from chapter 29 but this would not yield a structurization since σ is not at all simple.

In another application of ART, the First Recursion Theorems of Moschovakis for functionals and second order relations are improved in chapters 28 and 30, respectively.

Results established in chapters 23 and 26 suggest that the mathematical foundations of Computer Science provide another perspective field for conceptual and practical applications. ART is closely related to such topics as semantics of programming languages, structured and functional programming, proving program correctness etc.

There is little doubt that algebraization brings new coherence and insight to Recursion Theory. In chapter 9 we obtained new proofs of basic classical theorems, establishing a direct connection between results of Ordinary and Generalized Recursion Theories which turn out to be not mere analogues but particular instances of more general results of ART. The new concepts of ART also help to better understand classical ones. For example, the algebraic treatment of the transition property and the least fixed point operator respectively as translation and iteration in consecutive spaces is new both in Ordinary and Generalized Recursion Theories. The storing operation throws more light on the nature of certain technicalities of Generalized Recursion Theory, Inductive Definability Theory and Computer Science, contributing also to the 'lightface–boldface' division in the general theory itself. Splitting has also fully established its importance.

And now, ART and effective computability; we have referred to our starting point in chapter 2, presumably with a deeper knowledge. The motivation adduced in chapters 2, 12, 13 and the practical work carried out in parts E, F lead to the thesis that all natural concepts of effective computability can be formulated within suitable IOS, at worst having to use consecutive spaces. Therefore, the comparison with abstract IOS-notions of effective computability such as recursiveness, t-recursiveness and $\mathscr{B}'$-recursiveness provides an objective criterion for judging the naturalness of a proposed notion of effective computability and aids the choice between several such notions.

To make practical use of the above thesis one should be able to identify those mathematical notions which are notions of effective computability. (They need not be necessarily so called.) This can be done either by their intuitive content or by their effectivity properties such as transition, recursion and enumeration.

While Church's thesis and its various extensions identify the widest intuitively acceptable notion of effective computability in a given context, it could be claimed that narrower notions are in a sense more basic. For one can describe relative partial recursiveness in terms of relative μ-recursiveness but not vice versa; prime computability similarly embraces the wider notions of Friedman's computability and search computability. This underlines once again the fundamentality of the initial elements and operations of IOS which seem to form a minimal collection.

In addition to structurization, a principal problem of effective computability partly solved so far is the characterization of all those collections of operations $\mathscr{B}'$ and elements $\mathscr{B}_0$ which generate reasonable notions of effective computability $\mathscr{U} = cl(\mathscr{B}_0/\mathscr{B}')$. Such a notion should in particular satisfy the recursion property, which means that all unary mappings constructed by composing operations from $\mathscr{B}'$ and using constants from $\mathscr{B}_0$ have fixed points in $\mathscr{U}$ which are least with respect to a certain partial order. A related question. When does a given notion $\mathscr{U}$ have structurizable extension's which satisfy the recursion property? It would be interesting in connection with these problems to characterize alternatively the operations over an arbitrary IOS which are (equivalent to) t-operations.

Broadly speaking, it is to be hoped that ART will help one analyse effective computability as Group Theory helps to analyse symmetry.

We complete our discussion by outlining some possible directions for further work.

The first such direction deals with recursion theory on structures more general than IOS; developments of this kind were initiated by Zashev's dissertation devoted to general recursion theory on combinatory incomplete applicative systems. These are partially ordered sets with several constants called combinators, multiplication and either union or pairing operation satisfying certain weaker associative and distributive laws. Among the systems in question is a generalized OS, partially ordered set $\mathscr{F}$ with monotonic operations $\circ$, Π and constants I, L, R, $A_1 - A_4$, $D_1 - D_4$ such that $\varphi I = \varphi$, $(\varphi\psi)\chi = \varphi(\psi(\chi A_1))$, $\varphi(\psi\chi) = (\varphi(\psi A_2))(\chi A_3)$, $\varphi\psi = (\varphi A_4)(\psi A_3)$, $(\varphi, \psi)\chi = D_1(\varphi(\chi D_2), \psi(\chi D_3))$, $(\varphi\chi, \psi\chi) = ((\varphi, \psi)D_4)(\chi A_3)$, $L(\varphi, \psi) = \varphi$ and $R(\varphi, \psi) = \psi$. All the OS satisfy these axioms with $A_1 = \cdots = D_4 = I$. To get an example which is not an OS take an infinite set M such that $M^2 = M$, then take $\mathscr{F} = \{\varphi/\varphi \subseteq M\}$ and $\varphi\psi = \{s/\exists t \in \varphi((s, t) \in \psi)\}$. A further introduction to the subject can be found in Zashev [1983–198?].

In another development Skordev combinatory spaces are modified in Petrov and Skordev [1979] by considering categories instead of semigroups; the same can be done with IOS. So far, however, such generalizations have been less successful.

On the other hand, one may find even IOS too general a system and try to distinguish classes of models by means of additional axioms. For instance, 'multiple-valued' spaces are distinguished by a constant U such that $L, R \leq U$. (The element U itself need not necessarily be 'multiple-valued'; in the space of exercise 19.4 it is a single-valued function.) The availability of a second counter is expressed by making use of the constants W, W_1, W_2 in chapter 21.

A more ambitious task would be to axiomatize (parts of) particular recursion theories, turning to their more delicate and specific problems hitherto untouched by ART. Since consecutive spaces will be needed in some cases, the following remark is in order. The exposition of chapters 12–14 is not properly axiomatic but can be made so by introducing the notion of *higher* IOS which is an IOS $\mathscr{S}' = (\mathscr{F}', I', \Pi', L', R')$ with constants Id, $Ml \in \mathscr{F}'$ such that $I'^{\wedge} = I'$, $L'^{\wedge} = L'$, $R'^{\wedge} = R'$, $Id\varphi' = Id$, $Ml = Ml(L', R')$, $Ml(I', Id) = I'$, $Ml((L', L'R'), R'^2)$, $\langle\varphi'\rangle = (R'[(Ml(L', L'Id), Ml(R', R'Id))]\varphi')\check{}$, $Ml((L', L'R'), R'^2)$, $\langle\varphi'^{\wedge}\rangle = (R'[(Ml(L', L'Id), Ml(R', R'Id))]\varphi')^{\wedge}$, $[\varphi'^{\wedge}] = (R'[(Id, Ml)]\varphi')^{\wedge}$ and $\forall\theta'(\varphi'\theta' Id \leq \psi'\theta' Id) \Rightarrow \varphi' \leq \psi'$, $\varphi'^{\wedge}$ standing for $Ml(\varphi' Id, I')$. If $\mathscr{S}$, $\mathscr{S}'$ are consecutive IOS, then $\mathscr{S}'$ is a higher IOS and its subspace $\mathscr{S}'^{\wedge}$ based on $\{\varphi'^{\wedge}/\varphi' \in \mathscr{F}'\}$ is just the isomorphic copy $\bar{\mathscr{S}}$ of $\mathscr{S}$.

Conversely, let $\mathscr{S}'$ be a higher *IOS*. Assign to each $\varphi' \in \mathscr{F}'$ a mapping $\varphi'^{\cup} = \lambda\theta'^{\wedge}.(\varphi'\theta')^{\wedge}$ and take the isomorphic copy $\mathscr{S}'^{\cup}$ of $\mathscr{S}'$ based on $\{\varphi'^{\cup}/\varphi' \in \mathscr{F}'\}$. Then $\mathscr{S}'^{\wedge}, \mathscr{S}'^{\cup}$ are consecutive IOS.

The theory of consecutive spaces $\mathscr{S}$, $\mathscr{S}'$ of which the former is a pairing space and the latter an IOS (as in example 28.1) is similarly axiomatized by merely adding to the IOS-axioms the last axiom listed above with O' substituted for *Id*.

Another important direction for further work is the development of an axiomatic degree theory on IOS. (We regard it as a fascinating challenge of its own rather than a necessary part of the justification of the present approach.)

One way to introduce degrees in IOS is as follows. Consider the *adjunction operation* $\vee : \mathscr{F}^2 \to \mathscr{F}$ satisfying the 'light face' axioms $O \vee \varphi = \varphi \vee O = \varphi$, $(\varphi \vee \psi) \vee \chi = \varphi \vee (\psi \vee \chi)$, $L \vee R = L$, $\alpha(\varphi \vee \psi) = \alpha\varphi \vee \alpha\psi$, $(\varphi \vee \psi)\alpha = \varphi\alpha \vee \psi\alpha$ and $(\varphi L \vee R)(\alpha, \psi) = \varphi\alpha \vee \psi$ for $\alpha = L, R$, assuming also $\langle I \rangle = I$ and $\varphi = \varphi RL \Rightarrow \varphi = O$. One takes $\varphi \vee \psi = \sup\{\varphi, \psi \restriction (M \backslash \mathrm{Dom}\,\varphi)\}$ in example 4.7 and similarly in other spaces; compare with Moschovakis [1977].

Writing φ^+ for $\varphi L \vee R$, let $\varphi \leq_{\mathrm{RE}} \psi$ iff φ is recursive in ψ^+, $\varphi \leq_{\mathrm{R}} \psi$ iff $\varphi^+ \leq_{\mathrm{RE}} \psi$, $\varphi =_{\mathrm{R}} \psi$ iff $\varphi \leq_{\mathrm{R}} \psi$ & $\psi \leq_{\mathrm{R}} \varphi$,

$$D = \{a \subseteq \mathscr{F}/a \neq \varnothing \;\&\; \forall \varphi\psi \in a(\varphi =_{\mathrm{R}} \psi)\},$$

$a \leq_{\mathrm{RE}} b$ iff $\exists \varphi \in a \exists \psi \in b(\varphi \leq_{\mathrm{RE}} \psi)$, and $a \leq b$ iff $\exists \varphi \in a \exists \psi \in b(\varphi \leq_{\mathrm{R}} \psi)$. (Recursiveness could be replaced by other abstract notions of effective computability, e.g. t-recursiveness and $\mathscr{B}'$-recursiveness.) Now it can be shown that for every *degree a* in D there is a degree a', the *jump of a*, such that $a < a' \leq_{\mathrm{RE}} a$ and whenever $b \leq_{\mathrm{RE}} a$, then $b \leq a'$. The proposed object of study is the upper semilattice $(D, \leq)$ with or without the jump operator. It is certain however that in order to establish abstract analogues of the basic Degree Theory results one will need to assume further axioms and, perhaps, invent new techniques. This also seems to be an occasion for the concept of 'finite', much advocated by Kreisel, to duly enter.

Apart from Degree Theory, the adjunction operation can be employed in building up abstract analogues of the arithmetical hierarchy of Ordinary Recursion Theory; cf. Ivanov [1985] for some details on these hierarchies and degrees and their relationship.

An obvious question suggested by the abovementioned analogy between IOS and groups is whether the former support an interesting theory of a classically algebraic kind beyond the Semigroup Theory. And, if the answer is positive, then what is the benefit for Recursion Theory?

To summarize, while the present state of ART shows convincingly the feasibility of some algebraization of Recursion Theory, it will fall to future works both in the pure and applied theory to supply a more definite idea of how far can all this actually go.

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