PART D

Constructing operative spaces

CHAPTER 16

Pairing spaces

In this chapter we introduce certain algebraic systems, 'semi-manufactures' from which one can construct OS. Roughly speaking, these systems are OS without multiplication. The necessity of considering such simpler algebraic systems is suggested by 12.1, where an OS $\mathscr{S}'$ was constructed from a given space $\mathscr{S}$ assumed (somewhat excessively) to be an OS.

Some preliminary notions. Let $\mathscr{F}$ be a nonempty partially ordered set, let $\mathscr{T}$ be a nonempty *right directed set* (i.e., a partially ordered set each two members of which have an upper bound) and let $t:\mathscr{F}\to\mathscr{T}$. Letters $\varphi,\psi,\chi,\theta,\tau$ will stand for members of $\mathscr{F}$ and a,b,c for members of $\mathscr{T}$. We call $t(\varphi)$ the *type of* φ and write $\mathscr{F}_a$ for $\{\varphi/\varphi\in\mathscr{F}\,\&\,t(\varphi)\leq a\}$. A mapping $\Gamma:\mathscr{F}^n\to\mathscr{F}$ *preserves types* iff $\Gamma(\mathscr{F}_a^n)\subseteq\mathscr{F}_a$ for all a, while Γ is *normal* if $\exists a\forall b\geq a$ $(\Gamma(\mathscr{F}_b^n)\subseteq\mathscr{F}_b)$. Clearly, any type preserving mapping is normal. An n-ary mapping Γ is $\mathscr{T}$*-continuous* iff whenever for all i, $1\leq i\leq n$, $\{\varphi_{i,m}\}_m$ is a chain (i.e., increasing countable sequence) in $\mathscr{F}_a$, $\varphi_i\in\mathscr{F}$ and $\varphi_i=\sup_m\varphi_{i,m}$, then $\sup_m\Gamma(\varphi_{1,m},\ldots,\varphi_{n,m})$ exists and equals $\Gamma(\varphi_1,\ldots,\varphi_n)$. Notice that all $\mathscr{T}$-continuous mappings are monotonic. If $\mathscr{T}$ is a singleton, then the $\mathscr{T}$-continuous mappings are exactly the continuous ones and all the mappings over $\mathscr{F}$ preserve types. ('Continuous' means countable continuous here.)

Let $\Pi:\mathscr{F}^2\to\mathscr{F}$ and $L',R':\mathscr{F}\to\mathscr{F}$. As usual, (φ,ψ) will stand for $\Pi(\varphi,\psi)$. The quadruple $\mathscr{S}=(\mathscr{F},\Pi,L',R')$ is a *pairing space* iff the following hold.

1. Π,L',R' are monotonic.
2. $L'((\varphi,\psi))=\varphi$, $R'((\varphi,\psi))=\psi$.

In other words, Π is a *pairing operation* and L',R' are its inverses. To exclude trivialities assume that $\mathscr{F}$ is not a singleton.

A pairing space $\mathscr{S}$ is *continuous* iff so are Π,L',R'.

The following two statements are immediate.

Proposition 16.1. Let $\mathscr{F}$ be a set with at least two distinct members, let $\Pi:\mathscr{F}^2\to\mathscr{F}$ be injective and let $L',R':\mathscr{F}\to\mathscr{F}$ be its inverses. Take $\varphi\leq\psi$ iff $\varphi=\psi$. Then $\mathscr{S}=(\mathscr{F},\Pi,L',R')$ is a continuous pairing space.

Proposition 16.2. Whenever $(\mathscr{F},I,\Pi,L,R)$ is an OS, then $(\mathscr{F},\Pi,\lambda\theta.L\theta,\lambda\theta.R\theta)$ is a pairing space.

We now give several constructions which yield pairing spaces.

Proposition 16.3. Let M be a nonempty set, let $f_1, f_2: M \to M$ be injective and suppose that $f_1(M) \cap f_2(M) = \emptyset$. Let E be a partially ordered set with at least two distinct members and let e be a fixed member of it. Take $\mathscr{F} = \{\varphi/\varphi: M \to E\}$, $\varphi \le \psi$ iff $\forall s(\varphi(s) \le \psi(s))$, $(\varphi, \psi)(f_1(s)) = \varphi(s)$, $(\varphi, \psi)(f_2(s)) = \psi(s)$ and $(\varphi, \psi)(s) = e$ otherwise, $L'(\varphi) = \lambda s.\varphi(f_1(s))$ and $R'(\varphi) = \lambda s.\varphi(f_2(s))$. Then $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ is a continuous pairing space.

Proof. We have

$$L'((\varphi, \psi)) = \lambda s.(\varphi, \psi)(f_1(s)) = \lambda s.\varphi(s) = \varphi$$

and similarly $R'((\varphi, \psi)) = \psi$.

We are going to show that L', R' are continuous with respect to least upper bounds of arbitrary subsets of $\mathscr{F}$, while Π is continuous with respect to least upper bounds of nonempty subsets of $\mathscr{F}$. But let us show first that whenever $\varphi \in \mathscr{F}$ and $\mathscr{H} \subseteq \mathscr{F}$, then $\varphi = \sup \mathscr{H}$ iff $\varphi(s) = \sup\{\theta(s)/\theta \in \mathscr{H}\}$ for all s.

Suppose that $\varphi = \sup \mathscr{H}$. Then $\theta(s) \le \varphi(s)$ for all s and $\theta \in \mathscr{H}$. Let $s \in M$, $d \in E$ and $\theta(s) \le d$ for all $\theta \in \mathscr{H}$. Taking $\tau(s) = d$ and $\tau(t) = \varphi(t)$ otherwise, we get $\theta \le \tau$ for all $\theta \in \mathscr{H}$, hence $\varphi \le \tau$, which implies $\varphi(s) \le d$. Therefore, $\varphi(s) = \sup\{\theta(s)/\theta \in \mathscr{H}\}$.

Conversely, suppose that $\varphi(s) = \sup\{\theta(s)/\theta \in \mathscr{H}\}$ for all s. Then $\theta \le \varphi$ for all $\theta \in \mathscr{H}$. If $\theta \le \tau$ for all $\theta \in \mathscr{H}$, then $\theta(s) \le \tau(s)$ for all $\theta \in \mathscr{H}$ and all s, hence $\varphi(s) \le \tau(s)$ for all s. Therefore, $\varphi \le \tau$, which gives $\varphi = \sup \mathscr{H}$.

Now let $\varphi = \sup \mathscr{H}$. Then we have

$$L'(\varphi)(s) = \varphi(f_1(s)) = \sup\{\theta(f_1(s))/\theta \in \mathscr{H}\} = \sup\{L'(\theta)(s)/\theta \in \mathscr{H}\}$$

for all s, hence $L'(\varphi) = \sup L'(\mathscr{H})$ and similarly $R'(\varphi) = \sup R'(\mathscr{H})$.

Let $\mathscr{H}_1, \mathscr{H}_2 \neq \emptyset$, $\varphi = \sup \mathscr{H}_1$ and $\psi = \sup \mathscr{H}_2$. Then

$$(\varphi, \psi)(f_1(s)) = \varphi(s) = \sup\{\theta(s)/\theta \in \mathscr{H}_1\} = \sup\{(\theta, \tau)(f_1(s))/\theta \in \mathscr{H}_1 \,\&\, \tau \in \mathscr{H}_2\}.$$

Similarly, $(\varphi, \psi)(f_2(s)) = \sup\{(\theta, \tau)(f_2(s))/\theta \in \mathscr{H}_1 \,\&\, \tau \in \mathscr{H}_2\}$, while $(\varphi, \psi)(s) = e = \sup\{(\theta, \tau)(s)/\theta \in \mathscr{H}_1 \,\&\, \tau \in \mathscr{H}_2\}$ otherwise. Therefore, $(\varphi, \psi)(s) = \sup\{(\theta, \tau)(s)/\theta \in \mathscr{H}_1 \,\&\, \tau \in \mathscr{H}_2\}$ for all s, hence $(\varphi, \psi) = \sup \Pi(\mathscr{H}_1, \mathscr{H}_2)$. The proof is complete.

Some remarks are in order here.

If M, f_1, f_2 satisfy the assumptions of 16.3, then f_1, f_2 is said to be a *splitting scheme* for M. A slightly more general notion of splitting scheme is considered in Ivanov [1980, 1980b]. Multiple-valued splitting could also be of some interest.

Splitting schemes have already been used to construct OS in examples 4.7, 4.8. As mentioned there, all infinite sets admit splitting schemes. However, this general assertion depends on the axiom of choice, which is not the case in some particular instances. On the other hand, no set augmented with a splitting scheme can be finite since the members of the sequence $\{f_2^n(f_1(s))\}$ are pairwisely distinct for all s.

If $M = \omega$, then splitting schemes for M can be naturally specified. For instance, one may take $f_1 = \lambda s.2s$, $f_2 = \lambda s.2s + 1$ as in example 3.1, or $f_1 = \lambda s.3s$, $f_2 = \lambda s.3s + 1$ as in example 4.3 etc.

The set $M = \omega^* = \bigcup_n \omega^n$ can also be augmented with a natural splitting scheme. Writing Λ for the empty sequence and omitting brackets and commas, one may take $f_1 = \lambda x.0x$, $f_2(\Lambda) = \Lambda$ and $f_2(nx) = n+1x$.

If ξ_0 is a limit ordinal and $M = \{\xi/\xi < \xi_0\}$, then one may take $f_1 = \lambda\xi.2\xi$, $f_2 = \lambda\xi.2\xi + 1$.

An arbitrary nonempty set M_0 can be extended to a wider one which admits natural splitting schemes, e.g. $M = \omega \times M_0$ or $M = \omega^* \times M_0$. If M_0 is finite, then another possible extension is $M = M_0 \cup \omega$.

More generally, whenever f_1, f_2 is a splitting scheme for M and N is a nonempty set, then $\lambda sx.(f_1(s), x)$, $\lambda sx.(f_2(s), x)$ is a splitting scheme for $M \times N$. It follows in particular that 16.3 holds with $M \times N$ playing the role of M.

Proposition 16.4. Let M, f_1, f_2, E, e be the same as in 16.3 and N be an arbitrary nonempty set. Take $\mathscr{F} = \{\varphi/\varphi: M \times N \to E\}$, $\varphi \leq \psi$ iff $\forall sx(\varphi(s,x) \leq \psi(s,x))$, $(\varphi,\psi)(f_1(s),x) = \varphi(s,x)$, $(\varphi,\psi)(f_2(s),x) = \psi(s,x)$ and $(\varphi,\psi)(s,x) = e$ otherwise, $L'(\varphi) = \lambda sx.\varphi(f_1(s),x)$ and $R'(\varphi) = \lambda sx.\varphi(f_2(s),x)$. Then $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ is a continuous pairing space.

The existence of a splitting scheme is the standard assumption we regard as necessary for the development of a computability theory over the object domain M. This requirement is essentially weaker than the existence of a computable pairing function for M assumed in Platek [1966], Moschovakis [1969], Fenstad [1980], Skordev [1980] and other works. Indeed, given a pairing function J and two fixed distinct members s_1, s_2 of M, then $f_1 = \lambda s.J(s_1, s)$ and $f_2 = \lambda s.J(s_2, s)$ is a splitting scheme for M.

Let M be the open interval of reals $(0, 1)$. Then $f_1 = \lambda s.s/2$, $f_2 = \lambda s.(s+1)/2$ is a splitting scheme for M. Notice that f_1, f_2 (and f_1^{-1}, f_2^{-1}) are continuous functions in the ordinary sense, while a well known theorem of Weierstrass rules out the possibility of a continuous pairing function for $(0, 1)$. This fact is not insignificant since an approximation of reals by means of rationals would only be justified if continuous functions are used in the computations.

Another advantage of splitting as compared with pairing is that, as mentioned above, whenever M has a splitting scheme and N is a nonempty set, then we obtain a natural splitting scheme for $M \times N$. (This property is shared by the so called restricted pairing to be introduced in chapter 21.)

In order to consider a 'boldface' computability with arbitrary members of M allowed as enumeration indices however, one has to have a pairing function for M. This is arranged in our general theory by means of operation St. (Cf. exercises 10.1, 10.9.)

Given an arbitrary nonempty set M, then the wider set $\omega \times M^* = \omega \times \bigcup_n M^n$ admits a natural pairing function. One may take

$$J((k, s_1, \ldots, s_m), (l, t_1, \ldots, t_n)) = (J_0(k, l, m), s_1, \ldots, s_m, t_1, \ldots, t_n),$$

where J_0 codes triples of natural numbers. Of course, this extension of M does not differ essentially from that of Moschovakis [1969]. A common disadvantage of the extensions of all kinds is that one is interested in functions (relations etc.) over the set originally given which have to be distinguished from the others.

The following general construction produces new pairing spaces from given ones.

Proposition 16.5. Let $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ be a pairing space. Take $\mathscr{F}' = \{\varphi'/\varphi': \mathscr{F} \to \mathscr{F}\}$, $\varphi' \leq \psi'$ iff $\forall\theta(\varphi'(\theta) \leq \psi'(\theta))$, $\Pi'(\varphi', \psi') = \lambda\theta.(\varphi'(\theta), \psi'(\theta))$, $L''(\varphi') = \lambda\theta. L'(\varphi'(\theta))$ and $R''(\varphi') = \lambda\theta. R'(\varphi'(\theta))$. Then $\mathscr{S}' = (\mathscr{F}', \Pi', L'', R'')$ is a pairing space. Writing $\check{\varphi}$ for $\lambda\theta.\varphi$, the structure $\check{\mathscr{S}} = (\check{\mathscr{F}}, \Pi' \restriction \check{\mathscr{F}}^2, L'' \restriction \check{\mathscr{F}}, R'' \restriction \check{\mathscr{F}})$ is a subspace of $\mathscr{S}'$ isomorphic to $\mathscr{S}$.

Proof. If $\varphi' \leq \psi'$, then

$$L''(\varphi')(\theta) = L'(\varphi'(\theta)) \leq L'(\psi'(\theta)) = L''(\psi')(\theta)$$

for all θ, hence $L''(\varphi') \leq L''(\psi')$. Therefore, L'' is monotonic. The monotonicity of R'', Π' is verified similarly. It follows that

$$L''((\varphi', \psi'))(\theta) = L'((\varphi'(\theta), \psi'(\theta))) = \varphi'(\theta)$$

for all θ; hence $L''((\varphi', \psi')) = \varphi'$. Similarly, $R''((\varphi', \psi')) = \psi'$; hence $\mathscr{S}'$ is a pairing space. $\check{\mathscr{S}}$ and $\mathscr{S}$ are isomophic since $\check{\varphi} \leq \check{\psi}$ iff $\varphi \leq \psi$, $(\check{\varphi}, \check{\psi}) = (\varphi, \psi)\check{}$, $L''(\check{\varphi}) = L'(\varphi)\check{}$ and $R''(\check{\varphi}) = R'(\varphi)\check{}$. The proof is complete.

Proposition 16.6. Proposition 16.5 holds with 'continuous pairing space' substituted for 'pairing space'.

Proof. The argument adduced in the proof of 16.3 applies here to ensure that if $\varphi' \in \mathscr{F}'$, $\mathscr{H}' \subseteq \mathscr{F}'$, then $\varphi' = \sup \mathscr{H}'$ iff $\varphi'(\theta) = \sup\{\theta'(\theta)/\theta' \in \mathscr{H}'\}$ for all θ.

Let $\{\varphi'_n\}$ be a chain in $\mathscr{F}'$ and $\varphi' = \sup_n \varphi'_n$. Then $\varphi'(\theta) = \sup_n \varphi'_n(\theta)$ for all θ; hence

$$L''(\varphi')(\theta) = L'(\varphi'(\theta)) = \sup_n L'(\varphi'_n(\theta)) = \sup_n L''(\varphi'_n)(\theta)$$

for all θ by the continuity of L'. Therefore, $L''(\varphi') = \sup_n L''(\varphi'_n)$. The continuity of R'', Π' is verified similarly. The proof is complete.

Pairing spaces will be used to construct OS, but not all the pairing spaces give iterative OS. We introduce certain special kinds of pairing spaces which do.

A pairing space $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ is said to be *$\mathscr{T}$-complete* ($\mathscr{T}$-CPS) iff the following hold.

1. Π, L', R' preserve types.
2. For any $a \in \mathscr{T}$ and any well ordered subset $\mathscr{H}$ of $\mathscr{F}_a$ there is $\varphi \in \mathscr{F}_a$ such that $\varphi = \sup \mathscr{H}$ in $\mathscr{F}$.

A $\mathscr{T}$-complete pairing space is *strongly $\mathscr{T}$-complete* ($\mathscr{T}$-SCPS) if whenever $a \in \mathscr{T}$, $\mathscr{H}$ is a well ordered subet of $\mathscr{F}_a$ and $\varphi = \sup \mathscr{H}$ in $\mathscr{F}$, then $L'(\varphi) = \sup L'(\mathscr{H})$, $R'(\varphi) = \sup R'(\mathscr{H})$.

A pairing space $\mathscr{S}$ is *$\mathscr{T}$-continuously complete* ($\mathscr{T}$-CCPS) if the following hold.

1. Π, L', R' are $\mathscr{T}$-continuous.
2. Π, L', R' preserve types.
3. For any a and any chain $\{\varphi_n\}$ in $\mathscr{F}_a$ there is $\varphi \in \mathscr{F}_a$ such that $\varphi = \sup_n \varphi_n$ in $\mathscr{F}$.

4. $\mathscr{F}$ has a least element O.

If $\mathscr{T}$ is a singleton, then we say *complete pairing space* (CPS), *strongly complete pairing space* (SCPS), *continuously complete pairing space* (CCPS) respectively for $\mathscr{T}$-CPS, $\mathscr{T}$-SCPS and $\mathscr{T}$-CCPS. Notice that all CCPS are also continuous pairing spaces.

Proposition 16.7. If $\mathscr{S}$ is a $\mathscr{T}$-CPS, then there is $O \in \mathscr{F}$ such that $O \leq \varphi$ for all φ and $t(O) \leq a$ for all a.

Proof. For all a the set $\varnothing$ is a well ordered subset of $\mathscr{F}_a$; hence there is an $O \in \mathscr{F}_a$ such that $O = \sup \varnothing$ in $\mathscr{F}$. Therefore, the element in question is one and the same in all $\mathscr{F}_a$; hence $O \leq \varphi$ for all φ and $t(O) \leq a$ for all a. The proof is complete.

We shall assume without loss of generality that the element O in the definition of $\mathscr{T}$-CCPS also satisfies the inequality $t(O) \leq a$ for all a. Otherwise one can embed $\mathscr{T}$ into an upper semilattice $\mathscr{T}'$ by the construction given in 16.11 below and then take $\mathscr{T}_1 = \{t(O) \vee a / a \in \mathscr{T}'\}$ and $t_1(\varphi) = t(O) \vee t(\varphi)$, thereby ensuring that $\mathscr{S}$ is a $\mathscr{T}_1$-CCPS and $t_1(O) \leq a$ for all $a \in \mathscr{T}_1$.

The pairing space of 16.1 is neither a $\mathscr{T}$-CPS nor a $\mathscr{T}$-CCPS since it lacks a least element O.

Proposition 16.8. Let $\mathscr{S}, E$ be the same as in 16.3, and suppose that E has a least member $\perp$ and all chains in E have least upper bounds. Then $\mathscr{S}$ is a CCPS.

Proof. The element $O = \lambda s. \perp$ is a least member of $\mathscr{F}$. By the proof of 16.3, it suffices to prove that all chains in $\mathscr{F}$ have least upper bounds.

Let $\{\varphi_n\}$ be a chain in $\mathscr{F}$. Take $\varphi = \lambda s. \sup_n \varphi_n(s)$. Then $\varphi = \sup_n \varphi_n$ by the proof of 16.3. The proof is complete.

Proposition 16.9. Let $\mathscr{S}, E$ be the same as in 16.3 and suppose that all well ordered subsets of E have least upper bounds. Then $\mathscr{S}$ is both a SCPS and a CCPS.

Proof. By the proof of 16.3, it suffices to show that all well ordered subsets of $\mathscr{F}$ have least upper bounds.

If $\mathscr{H}$ is a well ordered subset of $\mathscr{F}$, then $\{\theta(s)/\theta \in \mathscr{H}\}$ is a well ordered subset of E for all s. Taking $\varphi = \lambda s. \sup\{\theta(s)/\theta \in \mathscr{H}\}$, we get $\varphi = \sup \mathscr{H}$, which completes the proof.

Proposition 16.10. Let $\mathscr{S}, E$ be the same as in 16.4. If E has a least member and all chains in E have least upper bounds, then $\mathscr{S}$ is a CCPS. If all well ordered subsets of E have least upper bounds, then $\mathscr{S}$ is both a SCPS and a CCPS.

This follows from 16.8, 16.9.

The construction of 16.3 is quite general, providing a variety of examples. Multiplication can be directly introduced in some of them to give OS or IOS; such examples will be studied in detail in chapter 21. By way of illustration we sketch a few possibilities here.

Take $E = \{\perp, \top\}$, $\perp < \top$. Then the space $\mathscr{S}$ of 16.3 is both a SCPS and a CCPS by 16.9. The members of $\mathscr{F}$ may be regarded as subsets of M. Namely, interpreting $\varphi(s)=\perp$, $\varphi(s)=\top$ as $s\notin\varphi$, $s\in\varphi$, it follows that $\mathscr{F}=\{\varphi/\varphi\subseteq M\}$ and $\varphi\leq\psi$ iff $\varphi\subseteq\psi$. Otherwise, $\varphi(s)=\perp$ could be interpreted as $s\in\varphi$, so that $\varphi\leq\psi$ iff $\psi\subseteq\varphi$ in this case.

Take $N=M$ and E as above. Then the space $\mathscr{S}$ of 16.4 is both a SCPS and a CCPS by 16.10. The members of $\mathscr{F}$ may be regarded as subsets of M^2, i.e. as binary relations on M. This can again be done in two ways, with either $\varphi\leq\psi$ iff $\varphi\subseteq\psi$ or $\varphi\leq\psi$ iff $\psi\subseteq\varphi$. Moreover, the members of $\mathscr{F}$ may be regarded as partial multiple-valued functions over M in two ways, with $t\in\varphi(s)$ iff $(s,t)\in\varphi$ or $t\in\varphi(s)$ iff $(t,s)\in\varphi$ respectively.

One may substitute the interval $[0,1]$ for $\{\perp,\top\}$ above. Then the resulting spaces consist of relations which can be regarded either as fuzzy or probabilistic, depending on how a multiplication operation is introduced.

The following basic construction is an adaptation of 16.5 in the case of complete spaces.

Proposition 16.11. Let $\mathscr{S}=(\mathscr{F},\Pi,L',R')$ be a $\mathscr{T}$-CPS ($\mathscr{T}$-SCPS). Take $\mathscr{F}'=\{\varphi'/\varphi':\mathscr{F}\to\mathscr{F}\,\&\,\varphi' \text{ is normal}\}$, $\leq,\Pi',L'',R'',\check{\mathscr{S}}$ as in 16.5,

$$\mathscr{T}'=\{a'/\varnothing\subset a'\subseteq\mathscr{T}\,\&\,\forall ab(a\in a'\,\&\,a\leq b\Rightarrow b\in a')\},$$

$a'\leq b'$ iff $b'\subseteq a'$ and $t'(\varphi')=\{a/\forall b\geq a(\varphi'(\mathscr{F}_b)\subseteq\mathscr{F}_b)\}$. Then $\mathscr{S}'=(\mathscr{F}',\Pi',L'',R'')$ is a $\mathscr{T}'$-CPS (respectively, $\mathscr{T}'$-SCPS) and $\mathscr{S},\check{\mathscr{S}}$ are isomorphic.

Proof. We first consider $\mathscr{T}'$. Obviously, $\mathscr{T}'$ is a partially ordered set. Notice that $a'\leq b'$ iff $\forall b\in b'\exists a\in a'(a\leq b)$. Let $a',b'\in\mathscr{T}'$. If $a\in a'$, $b\in b'$ and $c\geq a,b$, then $c\in a'\cap b'$, hence $a'\cap b'\neq\varnothing$ and $a'\cap b'\in\mathscr{T}'$. Moreover, $a',b'\leq a'\cap b'$, hence $\mathscr{T}'$ is a right directed set. In fact, $\mathscr{T}'$ is an upper semilattice and $\mathscr{T}$ is embedded in $\mathscr{T}'$ by assigning to each $a\in\mathscr{T}$ the member $\{b/a\leq b\}$ of $\mathscr{T}'$.

Let $\varphi'\in\mathscr{F}'$. Then $t'(\varphi')\neq\varnothing$ and $a\in t'(\varphi')$, $a\leq b$ imply $b\in t'(\varphi')$; hence $t'(\varphi')\in\mathscr{T}'$.

Suppose that $\varphi'\in\mathscr{F}'$ and $a\in t'(\varphi')$. Let $b\geq a$ and $\theta\in\mathscr{F}_b$. The mapping φ' is normal, so that $\varphi'(\theta)\in\mathscr{F}_b$. The mapping L' preserves types; hence $L'(\varphi'(\theta))\in\mathscr{F}_b$, i.e. $L''(\varphi')(\theta)\in\mathscr{F}_b$. Therefore the mapping $L''(\varphi')$ is normal, hence $L''(\varphi')\in\mathscr{F}'$. The above argument gives also $t'(\varphi')\subseteq t'(L''(\varphi'))$, i.e. $t'(L''(\varphi'))\leq t'(\varphi')$; hence L'' preserves types. Similarly, $R'':\mathscr{F}'\to\mathscr{F}'$, $\Pi':\mathscr{F}'^2\to\mathscr{F}'$ and R'',Π' preserve types. The verification that $\mathscr{S}'$ is a pairing space follows the proof of 16.5.

Suppose now that $a'\in\mathscr{T}'$ and $\mathscr{H}'$ is a well ordered subset of $\mathscr{F}'_{a'}$. Let $\theta\in\mathscr{F}$, $a\in a'$ and $b\geq a,t(\theta)$. We have $a\in t'(\theta')$ and $\theta'(\theta)\in\mathscr{F}_b$ for all $\theta'\in\mathscr{H}'$ since $t'(\theta')\leq a'$. Therefore, $\{\theta'(\theta)/\theta'\in\mathscr{H}'\}$ is a well ordered subset of $\mathscr{F}_b$, so there is a $\tau_\theta\in\mathscr{F}_b$ such that $\tau_\theta=\sup\{\theta'(\theta)/\theta'\in\mathscr{H}'\}$ in $\mathscr{F}$. The mapping $\varphi'=\lambda\theta.\tau_\theta$ is normal and $a'\subseteq t'(\varphi')$; hence $\varphi'\in F'_{a'}$. It is immediate that $\varphi'=\sup\mathscr{H}'$ in $\mathscr{F}'$. Therefore, $\mathscr{S}'$ is a $\mathscr{T}'$-CPS.

Suppose that $\mathscr{S}$ is strongly $\mathscr{T}$-complete. If $\mathscr{H}'$ is a well ordered subset of $\mathscr{F}'_{a'}$ and φ' is the same as above, then

$$\begin{aligned}L''(\varphi')(\theta)=L'(\varphi'(\theta))&=L'(\sup\{\theta'(\theta)/\theta'\in\mathscr{H}'\})=\sup\{L'(\theta'(\theta))/\theta'\in\mathscr{H}'\}\\&=\sup\{L''(\theta')(\theta)/\theta'\in\mathscr{H}'\}\end{aligned}$$

for all θ, hence $L''(\varphi') = \sup L''(\mathscr{H}')$. Similarly, $R''(\varphi') = \sup R''(\mathscr{H}')$, hence $\mathscr{S}'$ is a $\mathscr{T}'$-SCPS.

The isomorphism of $\mathscr{S}, \check{\mathscr{S}}$ is established as in 16.5, noting that the mappings $\check{\varphi}$ are normal, so that $\check{\mathscr{F}} \subseteq \mathscr{F}'$. It also follows that $t'(\check{\varphi}) = \{a/t(\varphi) \leq a\}$ and whenever $\mathscr{H} \subseteq \mathscr{F}_a$ and $\varphi = \sup \mathscr{H}$, then $\check{\varphi} = \sup \check{\mathscr{H}}$ in $\mathscr{F}'$. This completes the proof.

It is worth mentioning that $\check{\mathscr{S}}$ is a proper subspace of $\mathscr{S}'$ since L', $R' \in \mathscr{F}' \setminus \check{\mathscr{F}}$.

Proposition 16.12. Proposition 16.11 holds with '$\mathscr{T}$-CCPS', '$\mathscr{T}'$-CCPS' substituted respectively for '$\mathscr{T}$-CPS', '$\mathscr{T}'$-CPS'.

The proof follows that of 16.11. The mappings $\check{O} = \lambda\theta . O$ is a least member of $\mathscr{F}'$.

EXERCISES TO CHAPTER 16

Exercise 16.1. Let $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ be a pairing space with a least element O. Take $\mathscr{F}' = \{\varphi'/\varnothing \subset \varphi' \subseteq \mathscr{F}\}$, $\varphi' \leq \psi'$ iff $\forall \varphi \in \varphi' \exists \psi \in \psi' (\varphi \leq \psi)$, $\Pi'(\varphi', \psi') = \Pi(\varphi', \psi')$, $L''(\varphi') = L'(\varphi')$ and $R''(\varphi') = R'(\varphi')$. Show that $\mathscr{S}' = (\mathscr{F}', \Pi', L'', R'')$ is both a SCPS and a CCPS.

Remarks. The relation $\leq$ is not antisymmetric, hence $\mathscr{F}'$ is a quasi-ordered rather than partially ordered set and should be factorized. One may construct $\mathscr{S}'$ in a slightly different manner, taking $\mathscr{F}' = \{\varphi'/\varnothing \subset \varphi' \subseteq \mathscr{F} \,\&\, \forall \varphi\psi(\varphi \leq \psi \,\&\, \psi \in \varphi' \Rightarrow \varphi \in \varphi')\}$, $\varphi' \leq \psi'$ iff $\varphi' \subseteq \psi'$; no factorization is needed then.

One advantage of the construction of exercise 16.1 is that it gives a complete space without requiring $\mathscr{S}$ to be complete. Take for instance the pairing space $\mathscr{S}_1$ obtained from the OS of example 3.1 by 16.2, then take its subspace $\mathscr{S}$ consisting of functions with finite domains. The pairing space $\mathscr{S}$ has a least element but is neither a CPS nor a CCPS.

On the other hand, the above construction has a drawback. While the given pairing space $\mathscr{S}$ is isomorphic with the subspace $\mathscr{S}_0$ of $\mathscr{S}'$ based on $\mathscr{F}_0 = \{\{\varphi\}/\varphi \in \mathscr{F}\}$, one can not expect that whenever $\mathscr{S}$ is a CPS (or CCPS), then $\mathscr{S}_0$ is a subspace of $\mathscr{S}'$ as a CPS (a CCPS). It may well happen that $\varphi = \sup_n \varphi_n$ and $\varphi \neq \varphi_n$ for all n, so that $\{\varphi\} \neq \sup_n \{\varphi_n\}$.

Exercise 16.2. Let N be a nonempty set and $\mathscr{S}_s$ be a $\mathscr{T}_s$-CPS (respectively, $\mathscr{T}_s$-SCPS, $\mathscr{T}_s$-CCPS) for all $s \in N$. Take $\mathscr{F} = \times_{s \in N} \mathscr{F}_s$, i.e.

$$\mathscr{F} = \{\varphi/\varphi : N \to \bigcup_{s \in N} \mathscr{F}_s \,\&\, \forall s(\varphi(s) \in \mathscr{F}_s)\},$$

$\varphi \leq \psi$ iff $\forall s(\varphi(s) \leq \psi(s))$, $(\varphi, \psi) = \lambda s . \Pi_s(\varphi(s), \psi(s))$, $L'(\varphi) = \lambda s . L'_s(\varphi(s))$, $R'(\varphi) = \lambda s . R'_s(\varphi(s))$, $\mathscr{T} = \times_{s \in N} \mathscr{T}_s$, $a \leq b$ iff $\forall s(a(s) \leq b(s))$, and $t(\varphi) = \lambda s . t_s(\varphi(s))$. Show that $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ is a $\mathscr{T}$-CPS (respectively, $\mathscr{T}$-SCPS, $\mathscr{T}$-CCPS). In particular, if $\mathscr{S}_s$ is a CPS (SCPS, CCPS) for all s, then so is $\mathscr{S}$.

Exercise 16.3. Let $\mathscr{S}$ be a pairing space and $(L'(\varphi), R'(\varphi)) \leq \varphi$ for all φ. Show that Π is continuous with respect to least upper bounds of nonempty subsets of $\mathscr{F}$.

Hint. Supposing $\varnothing \subset \mathscr{H}_1, \mathscr{H}_2 \subseteq \mathscr{F}, \varphi_1 = \sup \mathscr{H}_1, \varphi_2 = \sup \mathscr{H}_2$ and $(\theta, \tau) \le \varphi$ for all $\theta \in \mathscr{H}_1, \tau \in \mathscr{H}_2$, show that $\varphi_1 \le L'(\varphi), \varphi_2 \le R'(\varphi)$, which implies $(\varphi_1, \varphi_2) \le \varphi$.

Exercise 16.4. Let $\mathscr{S}$ be a pairing space and $(L'(\varphi), R'(\varphi)) \ge \varphi$ for all φ. Show that L', R' are continuous with respect to least upper bounds.

Hint. Supposing $\varphi = \sup \mathscr{H}$ and $L'(\theta) \le \tau$ for all $\theta \in \mathscr{H}$, get $\varphi \le (\tau, R'(\varphi))$, which implies $L'(\varphi) \le \tau$.

As a corollary to the last two exercises, whenever $\Pi(\mathscr{F}, \mathscr{F}) = \mathscr{F}$ and $\mathscr{S}$ is $\mathscr{T}$-complete, then it is also strongly and continuously $\mathscr{T}$-complete.

Exercise 16.5. Let $\mathscr{S}$ be the pairing space of 16.3 or 16.4. Prove the following assertions. If $f_1(M) \cup f_2(M) \subset M$, then $\forall \varphi((L'(\varphi), R'(\varphi)) \le \varphi)$ iff $e \le d$ for all $d \in E$, while $\forall \varphi((L'(\varphi), R'(\varphi)) \ge \varphi)$ iff $\forall d(e \ge d)$. Finally, $\Pi(\mathscr{F}, \mathscr{F}) = \mathscr{F}$ iff $f_1(M) \cup f_2(M) = M$.

Exercise 16.6. Let $\mathscr{S}$ be a $\mathscr{T}$-CPS ($\mathscr{T}$-SCPS, $\mathscr{T}$-CCPS) and $a \in \mathscr{T}$. Prove that $\mathscr{S}_a = (\mathscr{F}_a, \Pi \restriction \mathscr{F}_a^2, L' \restriction \mathscr{F}_a, R' \restriction \mathscr{F}_a)$ is a CPS (respectively SCPS, CCPS).

Exercise 16.7. Take $\mathscr{F} = \{\varphi / \varphi \subseteq \omega^2\} = \{\varphi / \varphi : \omega \to 2^\omega\}$, $\varphi \le \psi$ iff $\varphi \subseteq \psi$, $(\varphi, \psi)(3s) = \varphi(s)$, $(\varphi, \psi)(3s+1) = \psi(s)$, $(\varphi, \psi)(3s+2)\uparrow$, $L'(\varphi) = \lambda s . \varphi(3s)$ and $R'(\varphi) = \lambda s . \varphi(3s+1)$, if $\operatorname{Dom} \varphi \subset \omega$, and $L'(\varphi)$, $R'(\varphi) = \omega^2$ otherwise. Show that $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ is a CPS but neither a SCPS nor a CCPS.

Hint. Take a chain $\{\varphi_n\}$ of single-valued functions such that $\operatorname{Dom} \varphi_n \subset \omega$ for all n and $\bigcup_n \operatorname{Dom} \varphi_n = \omega$. Then $L'(\sup_n \varphi_n) \ne \sup_n L'(\varphi_n)$.

Exercise 16.8. Let $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ be the pairing space of 16.4 with $N = M^\omega$ and $E = \{\bot, \top\}$, $\bot < \top$. Take $\mathscr{F}_1 = \{\varphi \in \mathscr{F} / \exists m \forall k(\varphi(\{s_n\}) = \top \Rightarrow s_{m+k} = s_m)\}$, $\mathscr{T} = \omega$ and

$$t(\varphi) = \min \{m / \forall k(\varphi(\{s_n\}) = \top \Rightarrow s_{m+k} = s_m)\}.$$

show that while $\mathscr{S}_1 = (\mathscr{F}_1, \Pi \restriction \mathscr{F}_1^2, L' \restriction \mathscr{F}_1, R' \restriction \mathscr{F}_1)$ is both a $\mathscr{T}$-SCPS and a $\mathscr{T}$-CCPS, it is neither CPS nor CCPS.

Hint. Notice that $\varphi \le \psi$ implies $t(\varphi) \le t(\psi)$, so any chain $\{\varphi_n\}$ such that $t(\varphi_n) = n$ for all n will have no upper bound in $\mathscr{F}_1$.

CHAPTER 17

Hierarchies of pairing spaces

Starting with a $\mathscr{T}$-complete (strongly $\mathscr{T}$-complete, continuously $\mathscr{T}$-complete) pairing space $\mathscr{S}$, we construct in this chapter a whole hierarchy $\{\mathscr{S}_\xi\}$ of $\mathscr{S}_\xi$-complete (respectively, strongly $\mathscr{T}_\xi$-complete, continuously $\mathscr{T}_\xi$-complete) pairing spaces, obtaining new examples of such spaces. The construction employed will also be useful later when building hierarchies of IOS.

To begin with, we construct an associated hierarchy of type sets $\{\mathscr{T}_\xi\}$.

Proposition 17.1. Given a nonempty right directed set $\mathscr{T}$, for all ξ right directed sets $\mathscr{T}_\xi$ can be constructed such that $\mathscr{T}_0 = \mathscr{T}$, the set $\mathscr{T}_\xi$ is obtained from $\mathscr{T}_{\xi_1}$ by 16.11, if $\xi = \xi_1 + 1$, while $\mathscr{T}_\xi = \{(\eta, a)/\eta < \xi \,\&\, a \in \mathscr{T}_\eta\}$, if $\xi > 0$ is a limit, and if $\xi \leq \eta$, then the members of $\mathscr{T}_\xi$ are *represented* in $\mathscr{T}_\eta$ by members of $\mathscr{T}_\eta$ (by themselves, if $\xi = \eta$) in such a way that the following condition is satisfied.

$$(\eta)\ \left\{\begin{array}{l}\text{If } \eta_1 \leq \eta_2 \leq \eta,\, a \in \mathscr{T}_{\eta_1},\, b \in \mathscr{T}_{\eta_2},\, a', a'' \text{ represent } a \text{ respectively in } \mathscr{T}_{\eta_2}, \mathscr{T}_\eta \\ \text{and } b' \text{ represents } b \text{ in } \mathscr{T}_\eta, \text{ then } a' \leq b \text{ iff } a'' \leq b'.\end{array}\right.$$

Taking $\eta_1 = \eta_2 = \xi$ in particular, it follows that $\mathscr{T}_\xi$ is isomorphic to a partially ordered subset of $\mathscr{T}_\eta$, provided $\xi \leq \eta$.

Proof. By transfinite induction on ξ.

Take $\mathscr{T}_0 = \mathscr{T}$.

Suppose that $\xi > 0$, right directed sets $\mathscr{T}_\eta$ have been constructed for all $\eta < \xi$ and if $\eta_1 \leq \eta_2 \leq \eta$, then a member of $\mathscr{T}_{\eta_2}$ is assigned to each member of $\mathscr{T}_{\eta_1}$ in such a way that (η) holds. This is the induction assumption.

We consider two separate cases, depending on whether ξ is a successor or a limit ordinal.

1. $\xi = \xi_1 + 1$.

We obtain $\mathscr{T}_\xi$ from $\mathscr{T}_{\xi_1}$ by 16.11. A member a of $\mathscr{T}_{\xi_1}$ is represented in $\mathscr{T}_\xi$ by $a' = \{b/b \in \mathscr{T}_{\xi_1} \,\&\, a \leq b\}$. If $\eta < \xi_1$, $a \in \mathscr{T}_\eta$, a' represents a in $\mathscr{T}_{\xi_1}$ by the induction assumption and a'' represents a' in $\mathscr{T}_\xi$, take a'' to represent a in $\mathscr{T}_\xi$.

Suppose that $\eta_1 \leq \eta_2 < \xi$, $a \in \mathscr{T}_{\eta_1}$, $b \in \mathscr{T}_{\eta_2}$ and a' represents a in $\mathscr{T}_{\eta_2}$. Then $\eta_1 \leq \eta_2 \leq \xi_1$ and the induction assumption provides (ξ_1), hence $a' \leq b$ iff $a'' \leq b'$, where a'', b' represent a, b in $\mathscr{T}_{\xi_1}$. If a''', b'' represent a, b in $\mathscr{T}_\xi$, then $a'' \leq b'$ iff $a''' \leq b''$ by the definition of a''', b'' and the construction of $\mathscr{T}_\xi$. Therefore, condition (ξ) is satisfied. (The case of $\eta_2 = \xi$ is obvious.)

2. ξ is a limit ordinal.

Take $\mathscr{T}_\xi = \{(\eta, a)/\eta < \xi \& a \in \mathscr{T}_\eta\}$ and $(\eta, a) \le (\zeta, b)$ iff $\eta \le \zeta \& a' \le b$, where a' represents a in $\mathscr{T}_\zeta$. The relation $\le$ is obviously reflexive and antisymmetric.

Suppose that $(\eta_1, a_1) \le (\eta_2, a_2)$ and $(\eta_2, a_2) \le (\eta_3, a_3)$. Let a_1', a_1'' represent a_1 respectively in $\mathscr{T}_{\eta_2}$, $\mathscr{T}_{\eta_3}$ and a_2' represent a_2 in $\mathscr{T}_{\eta_3}$. It follows that $\eta_1 \le \eta_2 \le \eta_3$ and $a_1' \le a_2$, hence $a_1'' \le a_2'$ by (η_3). Then $a_2' \le a_3$ implies $a_1'' \le a_3$, hence $(\eta_1, a_1) \le (\eta_3, a_3)$. Therefore, $\le$ is transitive and $\mathscr{T}_\xi$ is a partially ordered set.

Let $(\eta, a), (\zeta, b) \in \mathscr{T}_\xi$. Take $\xi_1 = \max\{\eta, \zeta\}$, a', b' to represent a, b in $\mathscr{T}_{\xi_1}$, then take $c \in \mathscr{T}_{\xi_1}$ such that $c \ge a', b'$. It follows that $(\xi_1, c) \in \mathscr{T}_\xi$ and $(\eta, a), (\zeta, b) \le (\xi_1, c)$; hence $\mathscr{T}_\xi$ is right directed.

If $a \in \mathscr{T}_\eta$, $\eta < \xi$, then take (η, a) to represent a in $\mathscr{T}_\xi$. Let $\eta_1 \le \eta_2 < \xi$, $a \in \mathscr{T}_{\eta_1}$, $b \in \mathscr{T}_{\eta_2}$ and a' represent a in $\mathscr{T}_{\eta_2}$. Then $a' \le b$ iff $(\eta_1, a) \le (\eta_2, b)$ by the definition of $\le$ in $\mathscr{T}_\xi$. Therefore, condition (ξ) is satisfied. The proof is complete.

Proposition 17.2. Let $\mathscr{S}$ be a $\mathscr{T}$-CPS (respectively, $\mathscr{T}$-SCPS, $\mathscr{T}$-CCPS) and $\{\mathscr{T}_\xi\}$ be obtained from $\mathscr{T}$ by 17.1. Then for each ordinal ξ a $\mathscr{T}_\xi$-CPS ($\mathscr{T}_\xi$-SCPS, $\mathscr{T}_\xi$-CCPS) $\mathscr{S}_\xi$ can be constructed such that whenever $\xi < \eta$, $\mathscr{S}_\xi$ is a proper subspace of $\mathscr{S}_\eta$.

Proof. By transfinite induction on ξ.

Take $\mathscr{S}_0 = \mathscr{S}$ and $t_0 = t$.

Suppose that $\xi > 0$ and $\mathscr{T}_\eta$-CPS (respectively, $\mathscr{T}_\eta$-SCPS, $\mathscr{T}_\eta$-CCPS) $\mathscr{S}_\eta$ with type functions t_η have been constructed for all $\eta < \xi$. Whenever $\eta = \eta_1 + 1 < \xi$, let $\mathscr{S}_\eta$ be obtained from $\mathscr{S}_{\eta_1}$ by 16.11 and let the subspace $\check{\mathscr{S}}_{\eta_1}$ of $\mathscr{S}_\eta$ be identified with $\mathscr{S}_{\eta_1}$. Whenever $\eta < \xi$ is a limit ordinal, then let all $\mathscr{S}_\zeta, \zeta < \eta$ be subspaces of $\mathscr{S}_\eta$ and $\mathscr{F}_\eta = \bigcup_{\zeta < \eta} \mathscr{F}_\zeta$. If $\varphi \in \bigcup_{\eta < \xi} \mathscr{F}_\eta$, then the ordinal $r(\varphi) = \min\{\eta/\varphi \in \mathscr{F}_\eta\}$ (zero or successor) is called the *rank of* φ. Finally, whenever $\eta < \xi$ and $\zeta = r(\varphi) < \eta$, let $t_\zeta(\varphi)$ be represented in $\mathscr{T}_\eta$ by $t_\eta(\varphi)$. This is the induction assumption.

There are two possibilities.

1. $\xi = \xi_1 + 1$.

Obtain $\mathscr{S}_\xi$, t_ξ from $\mathscr{S}_{\xi_1}, t_{\xi_1}$ by 16.11.

Let $\varphi \in \mathscr{F}_\xi$ and $\zeta = r(\varphi) < \xi$. Then $\zeta \le \xi_1$, hence $t_{\xi_1}(\varphi)$ represents $t_\zeta(\varphi)$ in $\mathscr{T}_{\xi_1}$ by the induction assumption. The proof of 16.11 gives $t_\xi(\varphi) = \{a \in \mathscr{T}_{\xi_1} / t_{\xi_1}(\varphi) \le a\}$, which is exactly the member of $\mathscr{T}_\xi$ representing $t_{\xi_1}(\varphi)$ in $\mathscr{T}_\xi$. Therefore, $t_\xi(\varphi)$ represents $t_\zeta(\varphi)$ in $\mathscr{T}_\xi$. The remaining assertions of the induction clause follow from the proof of 16.11 (16.12).

2. ξ is a limit ordinal.

Take $\mathscr{F}_\xi = \bigcup_{\eta < \xi} \mathscr{F}_\eta$ and $t_\xi(\varphi) = (r(\varphi), t_{r(\varphi)}(\varphi))$ for all $\varphi \in \mathscr{F}_\xi$. Take $\varphi \le_\xi \psi$ iff $\varphi \le_\zeta \psi$, $\Pi_\xi(\varphi, \psi) = \Pi_\zeta(\varphi, \psi)$, writing simply $(\varphi, \psi)_\xi = (\varphi, \psi)_\zeta$, where $\zeta = \max\{r(\varphi), r(\psi)\}$, and $L'_\xi(\varphi) = L'_\zeta(\varphi)$, $R'_\xi(\varphi) = R'_\zeta(\varphi)$, where $\zeta = r(\varphi)$. It follows that whenever $\zeta \le \eta < \xi$, then $(\varphi, \psi)_\xi = (\varphi, \psi)_\eta$, $L'_\xi(\varphi) = L'_\eta(\varphi)$, $R'_\xi(\varphi) = R'_\eta(\varphi)$ and $\varphi \le_\xi \psi$ iff $\varphi \le_\eta \psi$. Therefore, the quadruple $\mathscr{S}_\xi = (\mathscr{F}_\xi, \Pi_\xi, L_\xi, R_\xi)$ is a pairing space since so are $\mathscr{S}_\eta$ for all $\eta < \xi$.

If $\zeta = r(\varphi) < \xi$, then $t_\xi(\varphi) = (\zeta, t_\zeta(\varphi))$ represents $t_\zeta(\varphi)$ by the proof of 17.1.

Let $t_\xi(\varphi) = (\eta, t_\eta(\varphi))$, $t_\xi(\psi) = (\zeta, t_\zeta(\psi))$, $(\xi_1, a) \in \mathscr{T}_\xi$ and $t_\xi(\varphi)$, $t_\xi(\psi) \le (\xi_1, a)$. Then $\eta, \zeta \le \xi_1$, $t_{\xi_1}(\varphi)$, $t_{\xi_1}(\psi)$ represent $t_\eta(\varphi)$, $t_\zeta(\psi)$ in $\mathscr{T}_{\xi_1}$ and $t_{\xi_1}(\varphi)$, $t_{\xi_1}(\psi) \le a$. It follows that $r(\varphi), r(\psi) \le \xi_1$, hence $\varphi, \psi \in \mathscr{F}_{\xi_1}$, so $(\varphi, \psi)_\xi \in \mathscr{F}_{\xi_1}$ and

$t_{\xi_1}((\varphi, \psi)_\xi) \leq a$ since Π_{ξ_1} preserves types. Let $\xi_2 = r((\varphi, \psi)_\xi)$. Then $\xi_2 \leq \xi_1$, $t_\xi((\varphi, \psi)_\xi) = (\xi_2, t_{\xi_2}((\varphi, \psi)_\xi))$ and $t_{\xi_1}((\varphi, \psi)_\xi)$ represents $t_{\xi_2}((\varphi, \psi)_\xi)$ in $\mathscr{T}_{\xi_1}$, hence $t_\xi((\varphi, \psi)_\xi) \leq (\xi_1, a)$. (Observe that $(\varphi, \psi)_\xi = (\varphi, \psi)_{\xi_1} = (\varphi, \psi)_{\xi_2}$.) Therefore, Π_ξ preserves types. Similarly, L'_ξ and R'_ξ preserve types.

Let $a \in \mathscr{T}_\xi$ and $\mathscr{H}$ be a well ordered subset of $\mathscr{F}_{\xi,a} = \{\varphi \in \mathscr{F}_\xi / t_\xi(\varphi) \leq a\}$. Then $a = (\eta, b)$ for certain $\eta < \xi$, $b \in \mathscr{T}_\eta$ and $t_\xi(\theta) \leq a$ implies $\theta \in \mathscr{F}_\eta$, $t_\eta(\theta) \leq b$ for all $\theta \in \mathscr{H}$. Therefore, $\mathscr{H}$ is a well ordered subset of $\mathscr{F}_{\eta,b}$. There is by the induction assumption a $\varphi \in \mathscr{F}_{\eta,b}$ such that $\varphi = \sup \mathscr{H}$ in $\mathscr{F}_\eta$. We get $t_\xi(\varphi) \leq a$ by condition (ξ) of 17.1, hence $\varphi \in \mathscr{F}_{\xi,a}$. It follows that $\theta \leq_\xi \varphi$ for all $\theta \in \mathscr{H}$. Suppose that $\tau \in \mathscr{F}_\xi$ and $\theta \leq_\xi \tau$ for all $\theta \in \mathscr{H}$. Taking $\zeta = \max\{\eta, r(\tau)\}$, we get $\zeta < \xi$, $\mathscr{H} \subseteq \mathscr{F}_\zeta$, $\tau \in \mathscr{F}_\zeta$ and $\theta \leq_\zeta \tau$ for all $\theta \in \mathscr{H}$. However, $\varphi = \sup \mathscr{H}$ in $\mathscr{F}_\zeta$ since $\mathscr{S}_\eta$ is a subspace of $\mathscr{S}_\zeta$ by the induction assumption, hence $\varphi \leq_\zeta \tau$ which implies $\varphi \leq_\xi \tau$. Therefore, $\varphi = \sup \mathscr{H}$ in $\mathscr{S}_\xi$. This argument implies that $\mathscr{S}_\xi$ is a $\mathscr{T}_\xi$-CPS and whenever $\eta < \xi$, then $\mathscr{S}_\eta$ is a subspace of $\mathscr{S}_\xi$.

Assuming $\mathscr{S}$ a $\mathscr{T}$-SCPS, we show in addition that $L'_\xi(\varphi) = \sup L'_\xi(\mathscr{H})$ in $\mathscr{F}_\xi$ using the fact that $L'_\eta(\varphi) = \sup L'_\eta(\mathscr{H})$ in $\mathscr{F}_\eta$, where $\mathscr{H}, \eta, \varphi$ are as above. The case of R'_ξ is treated similarly.

Finally, assume that $\mathscr{S}$ is a $\mathscr{T}$-CCPS. Let $\{\varphi_n\}$, $\{\psi_n\}$ be chains in $\mathscr{F}_{\xi,a}$, $a = (\eta, b)$, $\eta < \xi$ and $b \in \mathscr{T}_\eta$. Then $\{\varphi_n\}$, $\{\psi_n\}$ are chains in $\mathscr{F}_{\eta,b}$. There are by the induction assumption $\varphi, \psi \in \mathscr{F}_{\eta,b}$ such that $\varphi = \sup_n \varphi_n$, $\psi = \sup_n \psi_n$, $(\varphi, \psi)_n = \sup_n (\varphi_n, \psi_n)_\eta$, $L'_\eta(\varphi) = \sup_n L'_\eta(\varphi_n)$ and $R'_\eta(\varphi) = \sup_n R'_\eta(\varphi_n)$ in $\mathscr{F}_\eta$. Proceeding as above, we get $\varphi, \psi \in \mathscr{F}_{\xi,a}$ and $\varphi = \sup_n \varphi_n$, $\psi = \sup_n \psi_n$, $(\varphi, \psi)_\xi = \sup_n (\varphi_n, \psi_n)_\xi$, $L_\xi(\varphi) = \sup_n L'_\xi(\varphi_n)$, $R'_\xi(\varphi) = \sup_n R'_\xi(\varphi_n)$ in $\mathscr{F}_\xi$. The element $O \in \mathscr{F}$ satisfies the inequality $O \leq_\xi \varphi$ for all $\varphi \in \mathscr{F}_\xi$ since $O \leq_\eta \varphi$ for all $\varphi \in \mathscr{F}_\eta$, $\eta < \xi$. Therefore, $\mathscr{S}_\xi$ is a $\mathscr{T}_\xi$-CCPS, which completes the induction and the proof.

EXERCISE TO CHAPTER 17

Let $\mathscr{S}$ be a CPS let $\{\mathscr{S}_\xi\}$ be a hierarchy based on it constructed by 17.2. The following exercises show how the types arise in initial segments of the hierarchy; the same happens if one starts with a SCPS or a CCPS.

Exercise 17.1. Show that $\mathscr{S}_n$ is a CPS for all n.

Exercise 17.2. Show that $\mathscr{S}_\omega$ is a ω-CPS with a type function $t_\omega = r$.

Exercise 17.3. Show that $\mathscr{S}_{\omega+n+1}$ is a ω-CPS with a type function $t_{\omega+n+1}(\varphi) = \min\{m / \forall k \geq m (\varphi(\mathscr{F}_{\omega+n,k}) \subseteq \mathscr{F}_{\omega+n,k})\}$ for all n.

CHAPTER 18

Conditions sufficient for iterativeness

When constructing IOS one has to verify that a structure satisfies certain axioms. While the axioms of OS are verified more or less directly, it is not convenient to do so with the μ-axiom. Fortunately, there are simple sufficient conditions which ensure iterativeness. Several such conditions are studied in this chapter, including the case of spaces with t-operations.

We start with two μ-Induction Theorems asserting the existence of least fixed points with nice properties. These theorems originate from a well known fixed point result of Tarski [1955] and its continuous version in Kleene [1952], and also generalize corresponding theorems of Skordev [1980] by involving types.

Assume that a partially ordered set $\mathscr{F}$, a type set $\mathscr{T}$ and a type function $t:\mathscr{F} \to \mathscr{T}$ are given.

Proposition 18.1. For all $a \in \mathscr{T}$ suppose that all the well ordered subsets of $\mathscr{F}_a$ have least upper bounds in $\mathscr{F}$ which are in $\mathscr{F}_a$. Let $\Gamma:\mathscr{F} \to \mathscr{F}$ be monotonic and normal. Then Γ has a fixed point which is a member of all subsets $\mathscr{E}$ of $\mathscr{F}$ which are closed under Γ and such that $\mathscr{E} \cap \mathscr{F}_a$ is closed under least upper bounds of well ordered subsets for all $a \in \mathscr{T}$. In particular, the fixed point concerned is exactly $\mu\theta.\Gamma(\theta)$.

Proof. There is an $a \in \mathscr{T}$ such that $\Gamma(\mathscr{F}_a) \subseteq \mathscr{F}_a$. We construct by transfinite induction an increasing sequence $\{\theta_\xi\}$ in $\mathscr{F}_a$ such that $\theta_\xi = \sup\{\Gamma(\theta_\eta)/\eta < \xi\}$ for all ξ.

Suppose that for all $\eta < \xi$ a $\theta_\eta \in \mathscr{F}_a$ has been constructed such that $\{\theta_\zeta\}_{\zeta \leq \eta}$ is increasing. Then $\{\theta_\eta\}_{\eta < \xi}$ is an increasing sequence in $\mathscr{F}_a$, hence so is $\{\Gamma(\theta_\eta)/\eta < \xi\}$ since Γ is monotonic and $\Gamma(\mathscr{F}_a) \subseteq \mathscr{F}_a$. Therefore, $\theta_\xi = \sup\{\Gamma(\theta_\eta)/\eta < \xi\}$ exists and is in $\mathscr{F}_a$. Certainly, the sequence $\{\theta_\eta\}_{\eta \leq \xi}$ is increasing, which completes the induction step.

Notice that if $\theta_\zeta = \theta_{\zeta+1}$, then $\theta_\xi = \theta_\zeta$ for all $\xi \geq \zeta$. And such a ζ actually exists since $\theta_\zeta = \theta_{\zeta+1}$ whenever $\mathrm{Card}(\zeta) > \mathrm{Card}(\mathscr{F}_a)$. The element θ_ζ is the required fixed point of Γ. In fact

$$\Gamma(\theta_\zeta) = \sup\{\Gamma(\theta_\eta)/\eta \leq \zeta\} = \theta_{\zeta+1} = \theta_\zeta.$$

In order to show that $\theta_\zeta \in \mathscr{E}$, we prove by transfinite induction that $\theta_\xi \in \mathscr{E}$ for all ξ.

Suppose that $\theta_\eta \in \mathscr{E}$ for all $\eta < \xi$. Then $\Gamma(\theta_\eta) \in \mathscr{E}$ for all $\eta < \xi$ since $\mathscr{E}$ is closed under Γ, hence $\{\Gamma(\theta_\eta)/\eta < \xi\}$ is a well ordered subset of $\mathscr{E} \cap \mathscr{F}_a$, which implies $\theta_\xi \in \mathscr{E}$.

Finally, let $\Gamma(\tau) \leq \tau$. Then the set $\mathscr{E} = \{\theta/\theta \leq \tau\}$ is closed both under Γ and under least upper bounds, hence $\theta_\zeta \in \mathscr{E}$. Therefore, $\theta_\zeta = \mu\theta.\Gamma(\theta)$ and the proof is complete.

Instead of well ordered sets one may use in the formulation of 18.1 totally ordered sets, increasing transfinite sequences and the like.

Least fixed points θ_ζ are obtained at level $\zeta \leq \omega$ in the continuous case considered below.

Proposition 18.2. Let $\mathscr{F}$ have a least member O and suppose that for all $a \in \mathscr{T}$ all the chains in $\mathscr{F}_a$ have least upper bounds in $\mathscr{F}$ which are in $\mathscr{F}_a$. Let $\Gamma : \mathscr{F} \to \mathscr{F}$ be $\mathscr{T}$-continuous and normal. Then Γ has a fixed point which is a member of all subsets $\mathscr{E}$ of $\mathscr{F}$ closed under Γ and such that $O \in \mathscr{E}$ and $\mathscr{E} \cap \mathscr{F}_a$ is closed under least upper bounds of chains for all $a \in \mathscr{T}$. The least fixed point in question is exactly $\mu\theta.\Gamma(\theta)$.

Proof. There is an $a \in \mathscr{T}$ such that $O \in \mathscr{F}_a$ and $\Gamma(\mathscr{F}_a) \subseteq \mathscr{F}_a$. Take $\theta_n = \Gamma^n(O)$ for all n. It follows that $\theta_0 \leq \theta_1$, while $\theta_n \leq \theta_{n+1}$ implies

$$\theta_{n+1} = \Gamma(\theta_n) \leq \Gamma(\theta_{n+1}) = \theta_{n+2}$$

by the monotonicity of Γ; hence $\{\theta_n\}$ is a chain. Moreover, $\theta_0 \in \mathscr{F}_a$ and $\theta_n \in \mathscr{F}_a$ implies $\theta_{n+1} \in \mathscr{F}_a$. Therefore, $\{\theta_n\}$ is a chain in $\mathscr{F}_a$, hence $\sigma = \sup_n \theta_n$ exists and is in $\mathscr{F}_a$. The element σ is the required fixed point of Γ. Actually,

$$\Gamma(\sigma) = \Gamma\left(\sup_n \theta_n\right) = \sup_n \Gamma(\theta_n) = \sup_n \theta_{n+1} = \sigma$$

by the $\mathscr{T}$-continuity of Γ. Moreover, $\theta_n \in \mathscr{E}$ for all n since $\theta_0 = O \in \mathscr{E}$ and $\mathscr{E}$ is closed under Γ. Therefore, $\sigma = \sup_n \theta_n \in \mathscr{E}$.

Let $\Gamma(\tau) \leq \tau$. Taking $\mathscr{E} = \{\theta/\theta \leq \tau\}$, we get $\sigma \leq \tau$, hence $\sigma = \mu\theta.\Gamma(\theta)$. The proof is complete.

Assume that an OS $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ with a type set $\mathscr{T}$ and a type function $t : \mathscr{F} \to \mathscr{T}$ are given. Bearing in mind the above fixed point theorems and the μ-axioms of chapter 5, we introduce the following conditions intended to ensure iterativeness.

(*) The operations $\circ, \Pi$ preserve types. If $a \in \mathscr{T}$ and $\mathscr{H}$ is a well ordered subset of $\mathscr{F}_a$, then there is a $\varphi \in \mathscr{F}_a$ such that $\varphi\psi = \sup(\mathscr{H}\psi)$ for all ψ.

(**) The operations $\circ, \Pi$ preserve types. Whenever $a \in \mathscr{T}$ and $\mathscr{H}$ is a well ordered subset of $\mathscr{F}_a$, then there is a $\varphi \in \mathscr{F}_a$ such that $L\varphi = \sup L\mathscr{H}$, $R\varphi = \sup R\mathscr{H}$ and $\varphi\psi = \sup(\mathscr{H}\psi)$ for all ψ.

(***) The operations $\circ, \Pi$ preserve types. If $a \in \mathscr{T}$ and $\{\varphi_n\}$ is a chain in $\mathscr{F}_a$, then there is a $\varphi \in \mathscr{F}_a$ such that $\varphi\psi = \sup_n(\varphi_n\psi)$, $\psi\varphi = \sup_n \psi\varphi_n$ for all ψ and $(I, \varphi) = \sup_n(I, \varphi_n)$. There is an element O such that $O \leq \psi$ and $O\psi = O$ for all ψ.

In view of the remark following 16.7 we may assume without loss of generality that $t(I)$, $t(L)$, $t(R)$, $t(O) \leq a$ for all a.

The following frequently used sufficient conditions are particular instances of the above ones, with $\mathcal{T}$ a singleton.

$(*)_0$ If $\mathcal{H}$ is a well ordered subset of $\mathcal{F}$, then there is a φ such that $\varphi\psi = \sup(\mathcal{H}\psi)$ for all ψ.

$(**)_0$ If $\mathcal{H}$ is a well ordered subset of $\mathcal{F}$, then there is a φ such that $L\varphi = \sup L\mathcal{H}$, $R\varphi = \sup R\mathcal{H}$ and $\varphi\psi = \sup(\mathcal{H}\psi)$ for all ψ.

$(***)_0$ If $\{\varphi_n\}$ is a chain in $\mathcal{F}$, then there is a φ such that $\varphi\psi = \sup_n(\varphi_n\psi)$, $\psi\varphi = \sup_n \psi\varphi_n$ for all ψ and $(I, \varphi) = \sup_n(I, \varphi_n)$. There is an element O such that $O \leq \psi$ and $O\psi = O$ for all ψ.

If $\mathcal{S}$ satisfies condition $(*)$ for certain $\mathcal{T}, t$, we shall say also that $\mathcal{S}$ is $(*)$-*complete*, and similarly for $(**)$ etc. The spaces of examples 3.1, 4.3, 4.7, 4.8 are $(**)_0$, $(***)_0$-complete, while that of example 3.2 is $(**)_0$-complete but not $(***)_0$-complete.

Proposition 18.3. Let $\mathcal{S}$, $\mathcal{T}$ satisfy $(**)$, $a \in \mathcal{T}$ and let $\mathcal{H}$ be a well ordered subset of $\mathcal{F}_a$. Then there is a $\varphi \in \mathcal{F}_a$ such that $\alpha\varphi\psi = \sup(\alpha\mathcal{H}\psi)$ for all $\alpha \in \mathcal{D}$, $\psi \in \mathcal{F}$.

This is immediate.

Proposition 18.4. Let $\mathcal{S}$, $\mathcal{T}$ satisfy $(***)$. Then the operations $\circ, \Pi$ are $\mathcal{T}$-continuous.

Proof. Suppose that $a \in \mathcal{T}$, $\{\varphi_n\}$, $\{\psi_n\}$ are chains in $\mathcal{F}_a$ and $\varphi = \sup_n \varphi_n$, $\psi = \sup_n \psi_n$.

It follows that $\varphi_n\psi_n \leq \varphi\psi$ for all n. Let $\tau \in \mathcal{F}$ and $\varphi_n\psi_n \leq \tau$ for all n. Then $\varphi_n\psi_m \leq \varphi_m\psi_m \leq \tau$ for all $m \geq n$, hence $\varphi_n\psi \leq \tau$ for all n by $(***)$, which implies $\varphi\psi \leq \tau$ by $(***)$. Therefore, $\varphi\psi = \sup_n \varphi_n\psi_n$.

Since $\{\varphi_n L\}$, $\{(I, \varphi_n L)\}$ are chains in $\mathcal{F}_a$, condition $(***)$ implies

$$(\varphi L, R) = (R, RL)(I, \varphi L) = A\left(I, \sup_n(\varphi_n L)\right) = A \sup_n (I, \varphi_n L)$$
$$= \sup_n A(I, \varphi_n L) = \sup_n (\varphi_n L, R).$$

Also $\{(\varphi_n L, R)\}$, $\{(I, \psi_n)\}$ are chains in $\mathcal{F}_a$, hence

$$(\varphi, \psi) = (\varphi L, R)(I, \psi) = \sup_n(\varphi_n L, R) \sup_n (I, \psi_n) = \sup_n(\varphi_n L, R)(I, \psi_n)$$
$$= \sup_n (\varphi_n, \psi_n).$$

The proof is complete.

Proposition 18.5. Let $\mathcal{S}, \mathcal{T}$ satisfy condition $(*)$ (condition $(**)$). Then $\mathcal{S}_1 = (\mathcal{F}, \Pi, \lambda\theta.L\theta, \lambda\theta.R\theta)$ is a $\mathcal{T}$-CPS ($\mathcal{T}$-SCPS respectively).

Proof. $\mathcal{S}_1$ is a pairing space by 16.2. It is $\mathcal{T}$-complete (strongly $\mathcal{T}$-complete) by $(*)$ (respectively, by $(**)$).

Proposition 18.6. Let $\mathscr{S}$, $\mathscr{T}$ satisfy (***) and let $\mathscr{S}_1$ be the same as in the previous statement. Then $\mathscr{S}_1$ is a $\mathscr{T}$-CCPS.

Proof. The pairing space $\mathscr{S}_1$ is continuously $\mathscr{T}$-complete by (***) and 18.4.

Proposition 18.7. Let $\mathscr{S}$, $\mathscr{T}$ satisfy (*) or (***). Then all inductive mappings over $\mathscr{F}$ preserve types.

Proof. Assume that $\mathscr{S}, \mathscr{T}$ satisfy (*).

The mappings $\lambda\theta_1\ldots\theta_n.\theta_i$, $1 \le i \le n$ and $\lambda\theta_1\ldots\theta_n.\psi$, $\psi\in\{I, L, R\}$ preserve types, of course.

If $\Gamma_1, \Gamma_2: \mathscr{F}^n \to \mathscr{F}$ preserve types, then so do

$$\Gamma = \lambda\theta_1\ldots\theta_n.\Gamma_1(\theta_1,\ldots,\theta_n)\Gamma_2(\theta_1,\ldots,\theta_n)$$

and

$$\Gamma = \lambda\theta_1\ldots\theta_n.(\Gamma_1(\theta_1,\ldots,\theta_n), \Gamma_2(\theta_1,\ldots,\theta_n))$$

since $\circ$, Π preserve types.

Suppose that $\Gamma_1: \mathscr{F}^{n+1} \to \mathscr{F}$ preserves types and

$$\Gamma = \lambda\theta_1\ldots\theta_n.\mu\theta.\Gamma_1(\theta_1,\ldots,\theta_n,\theta).$$

Let $\theta_1,\ldots,\theta_n \in \mathscr{F}_a$ and $\Gamma^* = \lambda\theta.\Gamma_1(\theta_1,\ldots,\theta_n,\theta)$. Then Γ^* is monotonic and normal and $\Gamma^*(\mathscr{F}_a) \subseteq \mathscr{F}_a$. If $\{\theta_n\}$ is the sequence assigned to Γ^* in the proof of 18.1, then $\theta_\xi \in \mathscr{F}_a$ for all ξ, hence $\mu\theta.\Gamma^*(\theta) \in \mathscr{F}_a$. Therefore, Γ preserves types.

If $\mathscr{S}, \mathscr{T}$ satisfy (***), then we make use of 18.2 instead of 18.1. The proof is complete.

Proposition 18.8. If $\mathscr{S}, \mathscr{T}$ satisfy (***), then all inductive mappings are $\mathscr{T}$-continuous.

Proof. Taking 18.4 into account, it suffices to show that whenever $\Gamma_1: \mathscr{F}^{n+1} \to \mathscr{F}$ is inductive and $\mathscr{T}$-continuous, then $\Gamma = \lambda\theta_1\ldots\theta_n.\mu\theta.\Gamma_1(\theta_1,\ldots,\theta_n,\theta)$ is $\mathscr{T}$-continuous.

Let $a \in \mathscr{T}$, $\{\varphi_{i,m}\}$ be a chain in $\mathscr{F}_a$ and $\varphi_i = \sup_m \varphi_{i,m}$ for all i, $1 \le i \le n$. Then the mapping Γ is monotonic and preserves types by 18.7, hence $\{\Gamma(\varphi_{1,m},\ldots,\varphi_{n,m})\}_m$ is a chain in $\mathscr{F}_a$. Therefore, $\sigma = \sup_m \Gamma(\varphi_{1,m},\ldots,\varphi_{n,m})$ exists by (***). It follows that $\Gamma(\varphi_{1,m},\ldots,\varphi_{n,m}) \le \Gamma(\varphi_1,\ldots,\varphi_n)$ for all m; hence $\sigma \le \Gamma(\varphi_1,\ldots,\varphi_n)$. On the other hand, the $\mathscr{T}$-continuity of Γ_1 implies

$$\begin{aligned}
&\Gamma_1(\varphi_1,\ldots,\varphi_n,\sigma)\\
&= \Gamma_1\left(\sup_m \varphi_{1,m},\ldots,\sup_m \varphi_{n,m}, \sup_m \Gamma(\varphi_{1,m},\ldots,\varphi_{n,m})\right)\\
&= \sup_m \Gamma_1(\varphi_{1,m},\ldots,\varphi_{n,m}, \Gamma(\varphi_{1,m},\ldots,\varphi_{n,m}))\\
&= \sup_m \Gamma(\varphi_{1,m},\ldots,\varphi_{n,m}) = \sigma,
\end{aligned}$$

hence $\Gamma(\varphi_1,\ldots,\varphi_n) \le \sigma$. Therefore, $\Gamma(\varphi_1,\ldots,\varphi_n) = \sigma$, hence Γ is $\mathscr{T}$-continuous. The proof is complete.

Proposition 18.9. Let $\mathcal{S}, \mathcal{T}$ satisfy $(*)$, let $\mathcal{E}$ be a normal segment and let $a \in \mathcal{T}$. Then $\mathcal{E} \cap \mathcal{F}_a$ is closed under least upper bounds of well ordered subsets.

Proof. Suppose that $\mathcal{A} \subseteq \mathcal{F}^2$,

$$\mathcal{E} = \{\theta / \theta\psi \leq \tau \quad \text{for all } \langle \psi, \tau \rangle \in \mathcal{A}\}$$

and $\mathcal{H}$ is a well ordered subset of $\mathcal{E} \cap \mathcal{F}_a$. Then $\theta\psi \leq \tau$ for all $\theta \in \mathcal{H}$, $\langle \psi, \tau \rangle \in \mathcal{A}$, hence $(\sup \mathcal{A})\psi \leq \tau$ for all $\langle \psi, \tau \rangle \in \mathcal{A}$ by $(*)$, i.e. $\sup \mathcal{H} \in \mathcal{E}$.

Proposition 18.10. Let $\mathcal{S}, \mathcal{T}$ satisfy $(**)$, let $\mathcal{E}$ be a regular segment and let $a \in \mathcal{T}$. Then $\mathcal{E} \cap \mathcal{F}_a$ is closed under least upper bounds of well ordered subsets.

The proof follows that of 18.9, using 18.3.

Scott , $O \in \mathcal{E}$

Proposition 18.11. Let $\mathcal{S}, \mathcal{T}$ satisfy $(***)$, let $\mathcal{E}$ be a regular segment and let $a \in \mathcal{T}$. Then $O \in \mathcal{E} \cap \mathcal{F}_a$ and $\mathcal{E} \cap \mathcal{F}_a$ is closed under least upper bounds of chains.

Proof. Suppose that $\mathcal{A} \subseteq \mathcal{D} \times \mathcal{F}^2$,

$$\mathcal{E} = \{\theta / \alpha\theta\psi \leq \tau \quad \text{for all } \langle \alpha, \psi, \tau \rangle \in \mathcal{A}\}$$

and $\{\varphi_n\}$ is chain in $\mathcal{E} \cap \mathcal{F}_a$. Then $\alpha\varphi_n\psi \leq \tau$ for all n and all $\langle \alpha, \psi, \tau \rangle \in \mathcal{A}$,
Cf 18.8 hence $\alpha(\sup_n \varphi_n)\psi \leq \tau$ for all $\langle \alpha, \psi, \tau \rangle \in \mathcal{A}$ by $(***)$, i.e. $\sup_n \varphi_n \in \mathcal{E}$. The proof of 6.5 implies $O \in \mathcal{E}$. This completes the proof.

Now we are ready to show that each of the above conditions implies iterativeness.

Proposition 18.12. If $\mathcal{S}$ is $(*)$-complete, then it is μA_2-iterative.

Proof. Let $\Gamma: \mathcal{F}^{n+1} \to \mathcal{F}$ be inductive and $\theta_1, \ldots, \theta_n \in \mathcal{F}$. Then the mapping $\Gamma^* = \lambda\theta . \Gamma(\theta_1, \ldots, \theta_n, \theta)$ is monotonic and normal since Γ preserves types by 18.7. It follows from 18.1, 18.9 that $\mu\theta . \Gamma^*(\theta)$ exists and is a member of all normal segments closed under Γ^*. Therefore, the axiom μA_2 holds, which completes the proof.

Proposition 18.13. If $\mathcal{S}$ is $(**)$-complete, then it is μA_3-iterative.

The proof follows that of 18.12, making use of 18.10 instead of 18.9.

μA_4

Proposition 18.14. If $\mathcal{S}$ is $(***)$-complete, then it is μA_3-iterative.

Proof. Let $\Gamma: \mathcal{F}^{n+1} \to \mathcal{F}$ be inductive and $\theta_1, \ldots, \theta_n \in \mathcal{F}$. Then the mapping $\Gamma^* = \lambda\theta . \Gamma(\theta_1, \ldots, \theta_n, \theta)$ is $\mathcal{T}$-continuous by 18.8 and normal by 18.7. It follows from 18.2, 18.11 that $\mu\theta . \Gamma^*(\theta)$ exists and is a member of all regular segments closed under Γ^*. The proof is complete.

The next five statements make it possible to transfer iterativeness from a given space to a related one; the last three of them are analogues to certain assertions of Skordev [1980].

Proposition 18.15. Let $\mathcal{S}$ be an OS, $\mathcal{S}_1$ be a subspace and suppose that $\mathcal{S}, \mathcal{T}$ satisfy condition $(*)$ (condition $(**)$). Let $\mathcal{F}_1$ be closed under least upper bounds of well ordered subsets of $\mathcal{F}_{1,a}$ for all $a \in \mathcal{T}$. Then $\mathcal{S}_1, \mathcal{T}$

satisfy $(*)$ (respectively, $(**)$) with a type function $t \upharpoonright \mathscr{F}_1$, and $\mathscr{S}_1$ is a subspace of $\mathscr{S}$ as an IOS.

This follows immediately.

Proposition 18.16. Let $\mathscr{S}$ be an OS, let $\mathscr{S}_1$ be a subspace and suppose that $\mathscr{S}, \mathscr{T}$ satisfy $(***)$. Let $O \in \mathscr{F}_1$ and let $\mathscr{F}_1$ be closed under least upper bounds of chains in $\mathscr{F}_{1,a}$ for all $a \in \mathscr{T}$. Then $\mathscr{S}_1, \mathscr{T}$ satisfy $(***)$ and $\mathscr{S}_1$ is a subspace of $\mathscr{S}$ as an IOS.

This follows immediately.

Proposition 18.17. Let $\mathscr{S}, \mathscr{T}$ satisfy $(*)$ and let $\mathscr{S}_1$ be obtained from $\mathscr{S}$ by introducing a new partial order $\leq_1$ such that whenever $a \in \mathscr{T}$, $\mathscr{H}$ is a subset of $\mathscr{F}_a$ well ordered with respect to $\leq$ and $\varphi = \sup \mathscr{H}$ with respect to $\leq$, then $\varphi = \sup \mathscr{H}$ with respect to $\leq_1$. Then $\mathscr{S}_1$ is μA_2-iterative and $\mathscr{S}, \mathscr{S}_1$ have identical operations $\langle\ \ \rangle$, $[\ \ ]$.

Proof. Notions related to the partial order $\leq_1$ will be subscripted, e.g. $\sup_1$, segment$_1$, $\mu_1 \theta . \Gamma(\theta)$ etc.

Let us prove first that whenever $\Gamma: \mathscr{F} \to \mathscr{F}$ is monotonic and normal, then $\mu\theta.\Gamma(\theta)$ is a member of all normal segments$_1$ closed under Γ. Given such a segment$_1$

$$\mathscr{E} = \{\theta / \theta\psi \leq_1 \tau \quad \text{for all } \langle \psi, \tau \rangle \in \mathscr{A}\},$$

$\mathscr{A} \subseteq \mathscr{F}^2$, take the sequence $\{\theta_\xi\}$ assigned to Γ by the proof of 18.1. It suffices to show that $\theta_\xi \in \mathscr{E}$ for all ξ.

Suppose that $\theta_\eta \in \mathscr{E}$ for all $\eta < \xi$. Then $\Gamma(\theta_\eta) \in \mathscr{E}$ for all $\eta < \xi$, hence $\Gamma(\theta_\eta)\psi \leq_1 \tau$ for all $\eta < \xi$ and all $\langle \psi, \tau \rangle \in \mathscr{A}$. We get $\theta_\xi \psi = \sup\{\Gamma(\theta_\eta)\psi / \eta < \xi\}$ by $(*)$, hence $\theta_\xi \psi = \sup_1 \{\Gamma(\theta_\eta)\psi / \eta < \xi\}$, which implies $\theta_\xi \psi \leq_1 \tau$ for all $\langle \psi, \tau \rangle \in \mathscr{A}$, i.e. $\theta_\xi \in \mathscr{E}$. Therefore, $\theta_\xi \in \mathscr{E}$ for all ξ, hence $\mu\theta.\Gamma(\theta) \in \mathscr{E}$.

If $\Gamma(\tau) \leq_1 \tau$, then $\mu\theta.\Gamma(\theta) \in \mathscr{E} = \{\theta / \theta \leq_1 \tau\}$. Therefore, $\mu_1\theta.\Gamma(\theta) = \mu\theta.\Gamma(\theta)$, and an inductive argument on the construction shows that a mapping is inductive$_1$ iff it is inductive. Taking 18.7 into account, we conclude that $\mathscr{S}_1$ is μA_2-iterative. This completes the proof.

Notice that whenever $\varphi \leq \psi$, then $\varphi \leq_1 \psi$ since $\psi = \sup\{\varphi, \psi\}$.

Proposition 18.18. Let $\mathscr{S}, \mathscr{T}$ satisfy $(**)$ and let $\mathscr{S}$, $\mathscr{S}_1$ be related as in the previous statement. Then $\mathscr{S}_1$ is μA_3-iterative and $\mathscr{S}, \mathscr{S}_1$ have identical operations $\langle\ \ \rangle$, $[\ \ ]$.

The proof follows that of 18.17.

Propositon 18.19. Let $\mathscr{S}, \mathscr{T}$ satisfy $(***)$ and let $\mathscr{S}_1$ be obtained from $\mathscr{S}$ by introducing a new partial order $\leq_1$ such that whenever $a \in \mathscr{T}$, $\{\varphi_n\}$ is a chain in $\mathscr{F}_a$ and $\varphi = \sup_n \varphi_n$, then $\varphi = \sup_{n1} \varphi_n$. Then $\mathscr{S}_1$ is μA_3-iterative and $\mathscr{S}, \mathscr{S}_1$ have identical operations $\langle\ \ \rangle$, $[\ \ ]$.

Proof. Following the proof of 18.17, we show first that whenever $\Gamma: \mathscr{F} \to \mathscr{F}$ is $\mathscr{T}$-continuous and normal, then $\mu\theta.\Gamma(\theta)$ is a member of all regular segments$_1$ $\mathscr{E}$ closed under Γ; we have $O \in \mathscr{E}$ since $\varphi \leq \psi$ always implies $\varphi \leq_1 \psi$.

Then we show that the inductive$_1$ mappings are exactly the inductive ones, which will complete the proof by 18.7, 18.8.

Assume that $\langle\ \ \rangle$ is a t-operation over $\mathscr{F}$ in the sense of chapter 10. (Mappings Σ_2, Σ_3 satisfying the equalities (2), (3) of chapter 10 need not be assumed to be given since their existence will follow by 10.9*, 10.10*.) The following sufficient conditions are designed to ensure that $\mathscr{S}$, $\langle\ \ \rangle$ satisfy a tμ-axion. Namely, take (t$*$) (respectively, (t$**$)) to be condition ($*$) (condition ($**$)) plus the additional assumption that $\langle\ \ \rangle$ preserves types and is continuous with respect to least upper bounds of well ordered subsets of $\mathscr{F}_a$ for all a. Take (t$***$) to be ($***$) plus the additional assumption that $\langle\ \ \rangle$ preserves types and is $\mathscr{T}$-continuous, and $\langle O\rangle = O$. (Notice that (t$*$), (t$**$) impose heavier constraints on the operation $\langle\ \ \rangle$ than (t$***$) does.)

The spaces and t-operations of exercise 10.1 satisfy both $(\mathrm{t}**)_0$, $(\mathrm{t}***)_0$, where the subscripts indicate that $\mathscr{T}$ is a singleton.

Proposition 18.20. If $\mathscr{S}, \langle\ \ \rangle, \mathscr{T}$ satisfy condition (t$*$) (condition (t$**$)), then $\mathscr{S}, \langle\ \ \rangle$ satisfy axiom tμA$_2$ (respectively, tμA$_3$).

Proof. $\mathscr{S}$ is μA$_2$-iterative by 18.12 (respectively, μA$_3$-iterative by 18.13). All t-inductive mappings are monotonic and preserve types. All t-simple segments satisfy the property established in 18.9. Therefore, $\mathscr{S}$, $\langle\ \ \rangle$ satisfy axiom tμA by 18.1. The proof is complete.

It often happens that all t-simple segments are normal by 10.18. Then one need not bother to verify that $\langle\ \ \rangle$ is continuous with respect to least upper bounds of well ordered subsets of $\mathscr{F}_a$, all a.

Proposition 18.21. Let $\mathscr{S}, \mathscr{T}$ satisfy condition ($*$) (condition ($**$)), let $\langle\ \ \rangle$ be monotonic and preserve types and let all the t-simple segments be normal (respectively regular). Then $\mathscr{S}, \langle\ \ \rangle$ satisfy axiom tμA$_2$(tμA$_3$).

Proof. $\mathscr{S}$ is μA$_2$-iterative by 18.12 (μA$_3$-iterative by 18.13). All t-inductive mappings are monotonic and preserve types; hence the validity of tμA follows by 18.9 (18.10) and 18.1.

Proposition 18.22. If $\mathscr{S}, \langle\ \ \rangle, \mathscr{T}$ satisfy (t$***$), then $\mathscr{S}, \langle\ \ \rangle$ satisfy the axiom tμA$_3$.

This follows from 18.2 since all t-inductive mappings preserve types and are $\mathscr{T}$-continuous in this case, while all t-simple initial segments satisfy the property of 18.11.

t-Analogues to 18.15–18.19 can be established as well.

EXERCISES TO CHAPTER 18

Exercise 18.1. Let $(L, R) \leq I$ hold in $\mathscr{S}$. Show that Π is continuous with respect to least upper bounds of nonempty subsets of $\mathscr{F}$. In particular, the requirement $(I, \varphi) = \sup_n (I, \varphi_n)$ in ($***$) could be dropped.

Hint. Use 16.2 and exercise 16.3.

Exercise 18.2. Let $(L,R) \geq I$ hold in $\mathscr{S}$. Show that whenever $\mathscr{H} \subseteq \mathscr{F}$ and $\varphi = \sup \mathscr{H}$, then $L\varphi = \sup L\mathscr{H}$ and $R\varphi = \sup R\mathscr{H}$. In particular, $(*)$ and $(**)$ will be equivalent.

Hint. Use 16.2 and exercise 16.4.

The equality $(L,R) = I$ is valid in examples 3.1, 3.2, while $(L,R) < I$ in example 4.3. As far as examples 4.7, 4.8 are concerned, everything depends on the splitting scheme.

Exercise 18.3. Let $\mathscr{S}$ be $(*)$-complete ($(**)$-complete, $(***)$-complete). Show that $\mathscr{S}_a = (\mathscr{F}_a, I, \Pi \restriction \mathscr{F}_a^2, L, R)$ is a $(*)_0$-complete (respectively, $(**)_0$-complete, $(***)_0$-complete) OS for all $a \in \mathscr{T}$.

Exercise 18.4. Let $\mathscr{S}$ be $(*)$-complete and Π $\mathscr{T}$-continuous. Observing that $\Delta(\varphi, \psi) = \mu\theta.(\varphi, \theta\psi)$ by 6.33, show that this least fixed point is reached at level ω and Δ is $\mathscr{T}$-continuous in its first argument. In particular, $\langle\ \ \rangle$ is $\mathscr{T}$-continuous.

Hint. Show that the mapping $\lambda\varphi\theta.(\varphi, \theta\psi)$ is $\mathscr{T}$-continuous, then use the proofs of 18.2, 18.7, 18.8.

Exercise 18.5. Let $\mathscr{S}$ be $(***)_0$-complete or $\mathscr{S}$ be $(*)_0$-complete and Π continuous. Prove that $\mathscr{S}$ admits a continuous collection operation. Assuming that $\mathscr{S}_1$ is obtained from $\mathscr{S}$ by modifying its partial order, show that $\mathscr{S}_1$ also admits a collection operation.

Hint. $\mathscr{S}$ satisfies condition $(c*)$ of exercise 11.1. The same operation Co does for $\mathscr{S}_1$.

Exercise 18.6. Let $\mathscr{S}$ be an OS, $\mathscr{S}_1$ a subspace, let $\mathscr{S}$ be $(***)$-complete and suppose that $\mathscr{F}_1$ be closed under $\langle\ \ \rangle$, $[\ \ ]$. Prove that $\mathscr{S}_1$ is μA_3-iterative and is a subspace of $\mathscr{S}$ as an IOS.

Hint. Show first that μA_3 holds for mappings $\Gamma: \mathscr{F}_1 \to \mathscr{F}_1$ recursive in $\mathscr{F}_1$, observing that $\mu_1\theta.\Gamma(\theta) = \sup_n \Gamma^n(O)$. Thus we obtain a First Recursion Theorem for $\mathscr{S}_1$, which in turn implies that μA_3 holds for arbitrary inductive mappings.

A version of the last exercise for spaces with t-operations can also be given.

Exercise 18.7. Let $\langle\!\langle\ \ \rangle\!\rangle$ be a storing operation which preserves types. Show that in order to ensure the validity of the stronger axiom $t\mu A$ of exercise 10.9 it suffices to require in $(*)$ and $(**)$ that $x\varphi = \sup x\mathscr{H}$ for all $x \in \mathscr{L}$, while condition $(t***)$ needs no amendments.

Exercise 18.8. Construct a 5-tuple $\mathscr{S}$ to meet μA_3 and all the axioms of OS but A_2.

Hint. Take $\mathscr{S}$ to be a $(**)_0$-complete OS with a pairing scheme modified as suggested by the hint to exercise 4.3. Then $\mathscr{S}$ will be $(**)_0$-complete, hence μA_3-iterative since the proof of 18.13 makes no use of A_2.

Exercise 18.9. Construct a 5-tuple $\mathscr{S}$ to meet μA_3 and all the axioms of OS but A_3.

Hint. Take $\mathscr{S}$ to be a $(***)_0$-complete OS with a pairing scheme modified as in the hint to exercise 4.3. The new pairing operation $\lambda\varphi\psi.\varphi$ is continuous, hence $\mathscr{S}$ is μA_3-iterative by the proof of 18.14.

While the OS of example 4.2 satisfies (£) but not (££), the following exercise shows that (£) is also independent.

Exercise 18.10. Construct an OS to satisfy (££) but not (£).

Hint. Take a μA_3-iterative OS $\mathscr{S}$ in which $\langle I\rangle = I$, then take $\mathscr{F}_1 = \{\varphi/\varphi$ is prime recursive$\}$. The OS $\mathscr{S}_1 = (\mathscr{F}_1, I, \Pi\restriction\mathscr{F}_1^2, L, R)$ satisfies (££) since $\mathscr{F}_1$ is closed under $[\ \ ]$. Assuming that the operation $\langle\ \ \rangle_1$ over $\mathscr{F}_1$ satisfies (£), get $\langle R\rangle = \langle R\rangle_1 \in \mathscr{F}_1$ contrary to exercise 6.14***.

CHAPTER 19

Constructing operative spaces

This chapter presents several standard constructions which produce OS from pairing spaces or partially ordered semigroups. In particular, consecutive spaces will be constructed.

Often a pairing space can be augmented with multiplication to become an OS.

Proposition 19.1. Let $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ be a pairing space, let $\circ : \mathscr{F}^2 \to \mathscr{F}$ be monotonic and associative and let $I \in \mathscr{F}$ be a unit such that $L' = \lambda\theta . L'(I)\theta$, $R' = \lambda\theta . R'(I)\theta$. Let $\circ$ be right distributive with respect to Π, i.e. $(\varphi, \psi)\chi = (\varphi\chi, \psi\chi)$ for all φ, ψ, χ. Then $\mathscr{S}_1 = (\mathscr{F}, I, \Pi, L'(I), R'(I))$ is an OS.

This follows from the relevant definitions.

Of course, the above statement gives no general construction since multiplication can sometimes be introduced in various ways. Once such an operation is singled out, 19.1 just says what has to be verified in order to that the structure at hand be an OS.

Proposition 19.2. Let $\mathscr{S}, \mathscr{S}_1$ be the same as in 19.1 and let $\mathscr{S}$ be $\mathscr{T}$-complete (strongly $\mathscr{T}$-complete). Let $\circ$ preserve types and suppose that whenever $a \in \mathscr{T}$, $\mathscr{H}$ is a well ordered subset of $\mathscr{F}_a$ and $\varphi = \sup \mathscr{H}$ in $\mathscr{F}$, then $\varphi\psi = \sup(\mathscr{H}\psi)$ for all ψ. Then $\mathscr{S}_1$ is $(*)$-complete ($(**)$-complete).

This follows from the corresponding definitions.

Proposition 19.3. Let $\mathscr{S}, \mathscr{S}_1$ be the same as in 19.1 and let $\mathscr{S}$ be continuously $\mathscr{T}$-complete. Let $\circ$ preserve types and be $\mathscr{T}$-continuous and suppose that O satisfies the equality $O\psi = O$ for all ψ. Then $\mathscr{S}_1$ is $(***)$-complete.

This follows from the corresponding definitions.

On the other hand, IOS can be obtained by augmenting certain partially ordered semigroups with pairing schemes.

Proposition 19.4. Let $\mathscr{F}$ be a partially ordered semigroup with a unit I and suppose that every subset $\mathscr{H}$ of $\mathscr{F}$ has a least upper bound φ such that $\varphi\psi = \sup(\mathscr{H}\psi)$ and $\psi\varphi = \sup \psi\mathscr{H}$ for all ψ. Let $L, L_1, R, R_1 \in \mathscr{F}$, $LL_1 = RR_1 = I$ and $LR_1 = RL_1 = O$, where $O = \sup \varnothing$. Take $(\varphi, \psi) = \sup\{L_1\varphi, R_1\psi\}$. Then $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is a $(**)_0$, $(***)_0$-complete OS.

Proof. Suppose that $\varnothing \subset \mathscr{H}_1, \mathscr{H}_2 \subseteq \mathscr{F}$, $\varphi = \sup \mathscr{H}_1$ and $\psi = \sup \mathscr{H}_2$.

Then

$$\begin{aligned}(\varphi,\psi) &= \sup\{L_1\varphi, R_1\psi\}\\ &= \sup\{\sup\{L_1\theta/\theta\in\mathscr{H}_1\},\quad \sup\{R_1\tau/\tau\in\mathscr{H}_2\}\}\\ &= \sup\{\sup\{L_1\theta, R_1\tau\}/\theta\in\mathscr{H}_1\&\tau\in\mathscr{H}_2\}\\ &= \sup\{(\theta,\tau)/\theta\in\mathscr{H}_1\&\tau\in\mathscr{H}_2\};\end{aligned}$$

hence Π is continuous with respect to least upper bounds of nonempty sets. In particular, Π is continuous and certainly monotonic. It follows that

$$\begin{aligned}&(\varphi,\psi)\psi = \sup\{L_1\varphi, R_1\psi\}\chi = \sup\{L_1\varphi\chi, R_1\psi\chi\} = (\varphi\chi,\psi\chi)\\ &L(\varphi,\psi) = L\sup\{L_1\varphi, R_1\psi\} = \sup\{LL_1\varphi, LR_1\psi\} = \sup\{\varphi, O\} = \varphi,\\ &R(\varphi,\psi) = \sup\{RL_1\varphi, RR_1\psi\} = \psi;\end{aligned}$$

hence $\mathscr{S}$ is a $(**)_0$-complete OS. Moreover, $O\leq\psi$ and $O\psi = (\sup\varnothing)\circ\psi = \sup\varnothing = O$ for all ψ, hence $\mathscr{S}$ is also $(***)_0$-complete. The proof is complete.

Notice that in this space we have also $\psi O = O$ for all ψ.

Proposition 19.5. (A typed version of 19.4.) Let $\mathscr{F}$ be a partially ordered semigroup with unit I such that $\circ$ preserves types and whenever $a\in\mathscr{T}$, $\mathscr{H}\subseteq\mathscr{F}_a$, there exists a $\varphi\in\mathscr{F}_a$ such that $\varphi\psi = \sup(\mathscr{H}\psi)$, $\psi\varphi = \sup\psi\mathscr{H}$ for all ψ. Let L, L_1, R, R_1, Π be as in 19.4. Then $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is a $(**)$, $(***)$-complete OS.

The proof repeats that of 19.4. If $\mathscr{H}\subseteq\mathscr{F}$ is finite, $\mathscr{H}\subseteq\mathscr{F}_a$ for a certain a; hence $\sup\mathscr{H}$ exists. Therefore, the operation Π is correctly introduced.

It is worth mentioning that the construction of 19.4, 19.5 applies equally well to the dual semigroup $(\mathscr{F}, \lambda\varphi\psi.\psi\varphi)$.

Proposition 19.6. Let $\mathscr{S}$ be constructed by 19.4 or 19.5. Then $\langle\varphi\rangle = \sup_n R_1^n L_1\varphi LR^n$, $\Delta(\varphi,\psi) = \sup_n R_1^n L_1\varphi\psi^n$ and $[\varphi] = \sup_n (R_1\varphi)^n L_1$. The element $U = \sup\{L, R\}$ satisfies condition (1) of exercise 7.10. The above characterizations of $\langle\ \rangle$, Δ, $[\ \ ]$ take place for all iterative subspaces of $\mathscr{S}$.

Proof. Proposition 6.33 gives by the proof of 18.2 that $\Delta(\varphi,\psi) = \sup_n\theta_n$, where $\theta_0 = O$, $\theta_{n+1} = (\varphi, \theta_n\psi)$. Supposing $\theta_n = \sup_{i<n} R_1^i L_1\varphi\psi^i$, one gets

$$\theta_{n+1} = \sup\{L_1\varphi, R_1\theta_n\psi\} = \sup_{i<n+1} R_1^i L_1\varphi\psi^i,$$

hence $\theta_n = \sup_{i<n} R_1^i L_1\varphi\psi^i$ for all n. Therefore, $\Delta(\varphi,\psi) = \sup_n R_1^n L_1\varphi\psi^n$. In particular, $\langle\varphi\rangle = \sup_n R_1^n L_1\varphi LR^n$ by 6.32.

It follows similarly that $[\varphi] = \sup_n\theta_n$, where $\theta_0 = O$ and $\theta_{n+1} = (I, \varphi\theta_n)$. One gets by induction that $\theta_n = \sup_{i<n}(R_1\varphi)^i L_1$ for all n, hence $[\varphi] = \sup_n (R_1\varphi)^n L_1$.

The element U satisfies condition (2) of exercise 7.10 since $\varphi U\psi = \varphi\sup\{L, R\}\psi = \sup\{\varphi L\psi, \varphi R\psi\}$ for all φ, ψ.

Suppose that $\mathscr{S}_1 = (\mathscr{F}_1, I, \Pi\restriction\mathscr{F}_1^2, L, R)$ is an iterative subspace of $\mathscr{S}$. Then $L_1, R_1\in\mathscr{F}_1$ since $L_1 = \sup\{L_1, R_1 O\} = (I, O)$ and $R_1 = (O, I)$. Therefore, the expressions for $\langle\ \ \rangle, \Delta, [\ \ ]$ obtained above still make sense. The proof is complete.

The following construction modifies that of 16.5.

Proposition 19.7. Let $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ be a pairing space (continuous pairing space). Take $\mathscr{F}' = \{\varphi'/\varphi' : \mathscr{F} \to \mathscr{F} \,\&\, \varphi'$ is monotonic (continuous)$\}$, $I' = \lambda\theta.\theta$, $\varphi' \leq \psi'$ iff $\forall\theta(\varphi'(\theta) \leq \psi'(\theta))$, $\varphi'\psi' = \lambda\theta.\varphi'(\psi'(\theta))$ and $\Pi'(\varphi', \psi') = \lambda\theta.(\varphi'(\theta), \psi'(\theta))$. Then $\mathscr{S}' = (\mathscr{F}', I', \Pi', L', R')$ is an OS.

Proof. If $\varphi', \psi' \in \mathscr{F}'$, then $\varphi'\psi' \in \mathscr{F}'$ since composition of mappings preserves monotonicity (continuity). It follows from the proof of 12.1 that $\mathscr{F}'$ is a partially ordered semigroup with unit I'. The monotonicity (respectively continuity) of L', R' implies $L', R' \in \mathscr{F}'$. If $\varphi', \psi' \in \mathscr{F}'$, then $(\varphi', \psi') \in \mathscr{F}'$ by the monotonicity (continuity) of Π. Moreover, Π' is monotonic since Π is. The distributive law is verified as in 12.1, while

$$L'(\varphi', \psi') = \lambda\theta.L'((\varphi'(\theta), \psi'(\theta))) = \lambda\theta.\varphi'(\theta) = \varphi'$$

and similarly $R'(\varphi', \psi') = \psi'$; hence $\mathscr{S}'$ is an OS. The proof is complete.

Notice that in the continuous version the operation multiplication and pairing of $\mathscr{S}'$ are continuous as well.

The following statement reaffirms the construction of 12.1.

Proposition 19.8. Let $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ be an OS and $\mathscr{S}' = (\mathscr{F}', I', \Pi', L', R')$ be obtained from the pairing space $(\mathscr{F}, \Pi, \bar{L}, \bar{R})$ by means of 19.7. Then $\mathscr{S}, \mathscr{S}'$ are consecutive OS.

Proof. We recall that $\bar{\varphi} = \lambda\theta.\varphi\theta$, $Id = \lambda\theta.I$ and $Ml = \lambda\theta.L\theta R\theta$. It follows that $\bar{\mathscr{F}} \subseteq \mathscr{F}'$ and $Id, Ml \in \mathscr{F}'$; hence the spaces $\mathscr{S}, \mathscr{S}'$ are consecutive.

The following statement modifies 16.11.

Proposition 19.9. Let $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ be a $\mathscr{T}$-CPS ($\mathscr{T}$-SCPS), $\mathscr{F}' = \{\varphi'/\varphi' : \mathscr{F} \to \mathscr{F} \,\&\, \varphi'$ is monotonic and normal$\}$, $I', \leq, \circ, \Pi'$ be introduced as in 19.7 and $\mathscr{T}', t'$ the same as in 16.11. Then $\mathscr{S}' = (\mathscr{F}', I', \Pi', L', R')$ is an OS and $\mathscr{S}', \mathscr{T}'$ satisfy $(*)$ (respectively $(**)$). In particular, if $\mathscr{S}$ is a CPS (SCPS), then $\mathscr{F}'$ consists of all the monotonic unary mappings over $\mathscr{F}$ and $\mathscr{S}'$ is $(*)_0$-complete ($(**)_0$-complete).

Proof. Suppose that φ', $\psi' \in \mathscr{F}'$, $a \in t'(\varphi')$ and $b \in t'(\psi')$. Let $c \geq a, b$ and $\theta \in \mathscr{F}_c$. The mapping ψ' is normal and $b \leq c$, hence $\psi'(\theta) \in \mathscr{F}_c$. The mapping φ' is normal and $a \leq c$, hence $\varphi'(\psi'(\theta)) \in \mathscr{F}_c$, i.e. $\varphi'\psi'(\theta) \in \mathscr{F}_c$. Therefore, $\varphi'\psi'$ is a monotonic and normal mapping, hence $\varphi'\psi' \in \mathscr{F}'$. Whenever $\varphi', \psi' \in \mathscr{F}'_{a'}$, then the above argument gives $a' \subseteq t'(\varphi'\psi')$, hence $\varphi'\psi' \in \mathscr{F}'_{a'}$. We conclude that the multiplication of $\mathscr{F}'$ preserves types. It follows similarly that $\mathscr{F}'$ is closed under Π' and Π' preserves types, using the fact that Π preserves types.

It is immediate that I' preserves types. Observing that L', R' preserve types, we get

$$t'(I') = t'(L') = t'(R') = \mathscr{T} \leq a'$$

for all $a' \in \mathscr{T}'$. In particular, I', L', R' are monotonic and normal, hence $I', L', R' \in \mathscr{F}'$. Using 19.7, it follows that $\mathscr{S}'$ is an OS.

Let $a' \in \mathscr{T}'$ and $\mathscr{H}'$ be a well ordered subset of $\mathscr{F}'_{a'}$. Then we take $\varphi' = \lambda\theta.\sup\{\theta'(\theta)/\theta' \in \mathscr{H}'\}$ as in the proof of 16.11 and show that φ' is monotonic and normal and also $a' \subseteq t'(\varphi')$, hence $\varphi' \in \mathscr{F}'_{a'}$. It follows that

$$\varphi'\psi'(\theta) = \varphi'(\psi'(\theta)) = \sup\{\theta'(\psi'(\theta))/\theta' \in \mathscr{H}'\} = \sup\{\theta'\psi'(\theta)/\theta' \in \mathscr{H}'\}$$

for all ψ', θ, hence $\varphi'\psi' = \sup(\mathscr{H}'\psi')$ for all ψ'. Therefore, $\mathscr{S}', \mathscr{T}'$ satisfy $(*)$.

Suppose that $\mathscr{S}$ is strongly $\mathscr{T}$-complete. Then

$$\begin{aligned} L'\varphi'(\theta) &= L'(\varphi'(\theta)) = L'(\sup\{\theta'(\theta)/\theta' \in \mathscr{H}'\}) = \sup\{L'(\theta'(\theta))/\theta' \in \mathscr{H}'\} \\ &= \sup\{L'\theta'(\theta)/\theta' \in \mathscr{H}'\} \end{aligned}$$

for all θ, hence $L'\varphi' = \sup L'\mathscr{H}'$. Similarly, $R'\varphi' = \sup R'\mathscr{H}'$, hence $\mathscr{S}', \mathscr{T}'$ satisfy $(**)$. The proof is complete.

Proposition 19.10. Let $\mathscr{S} = (\mathscr{F}, \Pi, L', R')$ be a $\mathscr{T}$-CCPS, $\mathscr{F}' = \{\varphi'/\varphi' : \mathscr{F} \to \mathscr{F} \;\&\; \varphi' \text{ is } \mathscr{T}\text{-continuous and normal}\}$, $I', \leq, \circ, \Pi'$ be as in 19.7 and $\mathscr{T}', t'$ those of 16.11. Then $\mathscr{S}' = (\mathscr{F}', I', \Pi', L', R')$ is an OS and $\mathscr{S}', \mathscr{T}'$ satisfy $(***)$. In particular, $\mathscr{F}'$ consists of all continuous unary mappings over $\mathscr{F}$ and $\mathscr{S}'$ is $(***)_0$-complete, provided $\mathscr{S}$ is a CCPS.

Proof. Suppose that $\varphi', \psi' \in \mathscr{F}'$, $a \in \mathscr{T}$, $\{\varphi_n\}$ is a chain in $\mathscr{F}_a$ and $\varphi = \sup_n \varphi_n$. The mapping ψ' is $\mathscr{T}$-continuous; hence $\psi'(\varphi) = \sup_n \psi'(\varphi_n)$. Let $b \in t'(\psi')$ and $c \geq a, b$. Then $\{\psi'(\varphi_n)\}$ is a chain in $\mathscr{F}_c$ since $b \leq c$ and $\varphi_n \in \mathscr{F}_c$ for all n. The mapping φ' is $\mathscr{T}$-continuous; hence $\varphi'(\psi'(\varphi)) = \sup_n \varphi'(\psi'(\varphi_n))$, i.e. $\varphi'\psi'(\varphi) = \sup_n \varphi'\psi'(\varphi_n)$. Therefore $\varphi'\psi'$ is a $\mathscr{T}$-continuous mapping. The proof of 19.9 implies that $\varphi'\psi'$ is normal, hence $\varphi'\psi' \in \mathscr{F}'$. The proof of 19.9 implies also that the multiplication of $\mathscr{F}'$ preserves types. It follows similarly that $\mathscr{F}'$ is closed under Π' and Π' preserves types on account of the fact that Π is $\mathscr{T}$-continuous and preserves types.

The mappings I', L', R' are $\mathscr{T}$-continuous and preserve types, hence $I, L', R' \in \mathscr{F}'$ and $t'(I'), t'(L'), t'(R') \leq a'$ for all $a' \in \mathscr{T}'$. Using 19.7, it follows that $\mathscr{S}'$ is an OS.

Suppose that $a' \in \mathscr{T}'$ and $\{\varphi'_n\}$ is a chain in $\mathscr{F}'_{a'}$. Let $\theta \in \mathscr{F}$, $a \in a'$ and $b \geq a, t(\theta)$. It follows that $\varphi'_n \in \mathscr{F}'_{a'}$; hence $a \in t'(\varphi'_n)$ and $\varphi'_n(\theta) \in \mathscr{F}_b$ for all n. Therefore, $\{\varphi'_n(\theta)\}$ is a chain in $\mathscr{F}_b$; hence there is a $\tau_\theta \in \mathscr{F}_b$ such that $\tau_\theta = \sup_n \varphi'_n(\theta)$. Take $\varphi' = \lambda\theta.\tau_\theta$. Then whenever $c \in \mathscr{T}$, $\{\varphi_n\}$ is a chain in $\mathscr{F}_c$ and $\varphi = \sup_n \varphi_n$, we obtain from the $\mathscr{T}$-continuity of φ'_n,

$$\begin{aligned} \varphi'(\varphi) &= \sup_n \varphi'_n(\varphi) = \sup_n \varphi'_n\left(\sup_m \varphi_m\right) = \sup_n \sup_m \varphi'_n(\varphi_m) \\ &= \sup_m \sup_n \varphi'_n(\varphi_m) = \sup_m \varphi'(\varphi_m); \end{aligned}$$

hence φ' is $\mathscr{T}$-continuous. It is also normal and $a' \subseteq t'(\varphi')$; hence $\varphi' \in \mathscr{F}'_{a'}$.

Using the $\mathscr{T}$-continuity of Π, we get

$$(I', \varphi')(\theta) = (\theta, \varphi'(\theta)) = \left(\theta, \sup_n \varphi'_n(\theta)\right) = \sup_n (\theta, \varphi'_n(\theta)) = \sup_n (I, \varphi'_n)(\theta)$$

for all θ; hence $(I', \varphi') = \sup_n (I', \varphi'_n)$.

If $\psi' \in \mathscr{F}'$, then

$$\varphi'\psi'(\theta) = \varphi'(\psi'(\theta)) = \sup_n \varphi'_n(\psi'(\theta)) = \sup_n \varphi'_n\psi'(\theta)$$

for all θ, hence $\varphi'\psi' = \sup_n(\varphi'_n\psi')$. It follows from the $\mathscr{T}$-continuity of ψ' that

$$\psi'\varphi'(\theta) = \psi'\left(\sup_n \varphi'_n(\theta)\right) = \sup_n \psi'(\varphi'_n(\theta)) = \sup_n \psi\varphi'_n(\theta)$$

for all θ; hence $\psi'\varphi' = \sup_n \psi'\varphi'_n$.

The mapping $O' = \check{O}$ is continuous and preserves types; hence $O' \in \mathscr{F}'$ and $t'(O') \leq a'$ for all a'. It follows that $O' \leq \psi'$ and $O'\psi' = \lambda\theta . O'(\psi'(\theta)) = \lambda\theta . O = O'$ for all ψ'. Therefore, $\mathscr{S}', \mathscr{T}'$ satisfy (***), which completes the proof.

The following two statements present our standard constructions yielding consecutive IOS. They will also be used in the next chapter to construct hierarchies of IOS.

Proposition 19.11. Let $\mathscr{S}$ be a (*)-complete ((**)-complete) OS and let $\mathscr{S}'$ be obtained from $(\mathscr{F}, \Pi, \bar{L}, \bar{R})$ by 19.9. Then $\mathscr{S}$, $\mathscr{S}'$ are consecutive IOS.

Proof. $(\mathscr{F}, \Pi, \bar{L}, \bar{R})$ is a $\mathscr{T}$-CPS ($\mathscr{T}$-SCPS) by 18.5, hence 19.9 may be applied. The spaces $\mathscr{S}, \mathscr{S}'$ are μA_2-iterative by 18.12 (respectively, μA_3-iterative by 18.13). All $\bar{\varphi}$ are monotonic and normal and so are Id, Ml; hence $\bar{\mathscr{F}} \subseteq \mathscr{F}'$, $Id, Ml \in \mathscr{F}'$. Therefore $\mathscr{S}, \mathscr{S}'$ are consecutive OS. In order to show that they are consecutive IOS it suffices to show that $\langle \bar{I} \rangle = \langle I' \rangle$. As mentioned in chapter 12 however, this equality follows by the μA_2-iterativeness of $\mathscr{S}$. The proof is complete.

Proposition 19.12. Let $\mathscr{S}$ be (***)-complete and let $\mathscr{S}'$ be obtained from $(\mathscr{F}, \Pi, \bar{L}, \bar{R})$ by 19.10. Then $\mathscr{S}, \mathscr{S}'$ are consecutive IOS.

Proof. $(\mathscr{F}, \Pi, \bar{L}, \bar{R})$ is a $\mathscr{T}$-CCPS by 18.6. Both $\mathscr{S}, \mathscr{S}'$ are μA_3-iterative by 18.14. The mappings $\bar{\varphi}, Id, Ml$ are $\mathscr{T}$-continuous and normal, hence $\bar{\mathscr{F}} \subseteq \mathscr{F}'$, $Id, Ml \in \mathscr{F}'$, which implies that $\mathscr{S}, \mathscr{S}'$ are consecutive OS. The equality $\langle \bar{I} \rangle = \langle I' \rangle$ also holds, hence $\mathscr{S}, \mathscr{S}'$ are consecutive IOS. The proof is complete.

Proposition 19.13. Let $\mathscr{S}'$ be obtained from $\mathscr{S}$ by 19.11. Then the monotonic mappings Q_σ of chapter 13 are members of $\mathscr{F}'$ and so are the mappings $Q_{\sigma,\mathscr{C}}$, provided L and R have an upper bound U.

This follows immediately, from the normality of these mappings.

If $\mathscr{S}'$ is obtained from $\mathscr{S}$ by 19.12 however, one should not expect that $Q_\sigma, Q_{\sigma,\mathscr{C}} \in \mathscr{F}'$ for arbitrary $\sigma, \mathscr{C}$. A counterexample will be given in the exercises. This indicates that the monotonic version is richer and perhaps, more interesting, than the continuous one.

The following statement shows that consecutive spaces constructed by 19.11 admit transfers.

Proposition 19.14. Let $\mathscr{S}, \mathscr{T}$ satisfy condition (*) (condition (**)), let $\mathscr{S}', \mathscr{T}'$ be obtained from $\mathscr{S}, \mathscr{T}$ by 19.11 and let $\mathscr{S}'', \mathscr{T}''$ be obtained from $\mathscr{S}', \mathscr{T}'$ by

19.11 again. Then

$$Tf = \lambda\varphi''.\lambda\theta'.\lambda\theta.\varphi''(\theta'(L'\breve{\theta}, I'))(R\theta)$$

is a correctly introduced operation over $\mathscr{F}''$ and the axiom tfμA$_2$ (respectively, tfμA$_3$) is valid. If $\mathscr{S}'''$ is obtained from $\mathscr{S}''$ by 19.11, then $Tf \in \mathscr{F}'''$.

Proof. Suppose that $\varphi'' \in \mathscr{F}'', \theta' \in \mathscr{F}', a' \in t''(\varphi'')$ and $b' \geq a', t'(\theta')$. Let $b \in b'$ and $\theta \in \mathscr{F}_b$. Then $t'(\breve{\theta}) \leq \breve{b} = \{a/b \leq a\}$ and $t'(\theta') \leq b' \leq \breve{b}$; hence $\theta'(L'\breve{\theta}, I') \in \mathscr{F}'_{\breve{b}}$. It follows that $\varphi''(\theta'(L'\breve{\theta}, I')) \in \mathscr{F}'_{\breve{b}}$ since $a' \leq \breve{b}$. Noting that $R\theta \in \mathscr{F}_b$ and $b \in \breve{b}$, we get $\varphi''(\theta'(L'\breve{\theta}, I'))(R\theta) \in \mathscr{F}_b$. Therefore, the monotonic mapping $Tf(\varphi'')(\theta') = \lambda\theta.\varphi''(\theta'(L'\breve{\theta}, I'))(R\theta)$ is normal, and hence in $\mathscr{F}'$.

Suppose that $\varphi'' \in \mathscr{F}''$ and $a' \in t''(\varphi'')$. If $\theta' \in \mathscr{F}'_{a'}$, then the above argument gives $t'(Tf(\varphi'')(\theta')) \leq a'$, i.e. $Tf(\varphi'')(\theta') \in \mathscr{F}'_{a'}$. Therefore, the monotonic mapping $Tf(\varphi'')$ over $\mathscr{F}'$ is normal; hence $Tf(\varphi'') \in \mathscr{F}''$. We conclude that $Tf: \mathscr{F}'' \to \mathscr{F}''$. Moreover, we get $Tf(\mathscr{F}''_{a''}) \subseteq \mathscr{F}''_{a''}$ for all a'', hence Tf preserves types. Propositions 10.18, 18.21 ensure that $\mathscr{S}''$, Tf satisfy tfμA$_2$ (respectively tfμA$_3$).

Finally, if $\mathscr{S}'''$ is obtained from $\mathscr{S}''$ by 19.11, then $Tf \in \mathscr{F}'''$ since Tf is monotonic and normal. This completes the proof.

One last remark. In order to consider the transfer operation over $\mathscr{F}''$, the space $\mathscr{S}$ need not be an OS. It suffices to assume $\mathscr{S}$ a pairing space and replace throughout chapter 14 $L\theta, R\theta$ by $L'(\theta), R'(\theta)$.

EXERCISES TO CHAPTER 19

Exercise 19.1. Let $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ be an OS with a least element O such that $O\psi = O$ for all ψ. Take $\mathscr{F}' = \{\varphi'/\varnothing \subset \varphi' \subseteq \mathscr{F}\}$, $\varphi' \leq \psi'$ iff $\forall\varphi \in \varphi' \exists \psi \in \psi'(\varphi \leq \psi)$, $\varphi'\psi' = \{\varphi\psi/\varphi \in \varphi' \,\&\, \psi \in \psi\}$, $\Pi'(\varphi', \psi') = \Pi(\varphi', \psi')$, $I' = \{I\}$, $L' = \{L\}$ and $R' = \{R\}$. Show that $\mathscr{S}' = (\mathscr{F}', I', \Pi', L', R')$ is a $(**)_0$, $(***)_0$-complete OS and the element $U' = \{L, R\}$ satisfies condition (1) of exercise 7.10. Give explicit characterizations of the operations $\langle\quad\rangle, \Delta, [\quad]$ of $\mathscr{S}'$.

Hint. Use exercise 16.1 and propositions 19.1–19.3.

Remarks. The necessity of factorization can be avoided by modifying $\mathscr{F}'$ as suggested in the remarks to exercise 16.1. The advantages and drawbacks of this construction are the same as in the case of pairing spaces. One always gets an iterative space, even if the given one is not; e.g. the subspace of an arbitrary IOS consisting of all elements polynomial in O, I is not iterative by exercise 6.14***. The given OS $\mathscr{S}$ is isomorphic with the subspace of $\mathscr{S}'$ based on $\{\{\varphi\}/\varphi \in \mathscr{F}\}$. If, however, $\mathscr{S}$ is iterative, this isomorphism may fail to agree with the operations $\langle\quad\rangle, [\quad]$ of $\mathscr{S}$ and $\mathscr{S}'$.

Exercise 19.2. Let N be a nonempty set and $\mathscr{S}_s$ be an IOS for all $s \in N$. Take $\mathscr{F} = \times_{s \in N} \mathscr{F}_s$, $\varphi \leqslant \psi$ iff $\forall s(\varphi(s) \leq \psi(s))$, $\varphi\psi = \lambda s.\varphi(s)\psi(s)$, $(\varphi, \psi) = \lambda s.\Pi_s(\varphi(s), \psi(s))$, $I = \lambda s.I_s$, $L = \lambda s.L_s$ and $R = \lambda s.R_s$. Show that $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is an IOS and whenever $\mathscr{S}_s, \mathscr{T}_s$ satisfy one of $(*)$, $(**)$, $(***)$ for all s, then $\mathscr{S}, \mathscr{T}$ satisfy the same condition, where $\mathscr{T}$ is the type set of exercise 16.2.

Exercise 19.3. Let $\mathscr{S}$ be an IOS, $\mathscr{F}^{\cup}$ a set and ${}^{\cup}: \mathscr{F} \to \mathscr{F}^{\cup}$ a bijection. Take

$\varphi^{\cup} \leq^{\cup} \psi^{\cup}$ iff $\varphi \leq \psi$, $\varphi^{\cup} \circ^{\cup} \psi^{\cup} = (\varphi\psi)^{\cup}$ etc. Show that $\mathcal{S}^{\cup} = (\mathcal{F}^{\cup}, I^{\cup}, \Pi^{\cup}, L^{\cup}, R^{\cup})$ is an IOS isomorphic with $\mathcal{S}$.

Remarks. It also follows that t-operations over $\mathcal{F}$ are transformed by $^{\cup}$ into t-operations over $\mathcal{F}^{\cup}$ etc. Despite its seeming triviality this construction has interesting applications, for if $\mathcal{F}^{\cup} = \mathcal{F}$, then the operation $^{\cup}$ need not necessarily be expressible by the initial IOS-operations of $\mathcal{S}$.

Exercise 19.4. Let $\mathcal{S}$ be the IOS of example 4.8, $\mathcal{F}^{\cup} = \mathcal{F}$ $\varphi^{\cup} = \varphi^{-1} = \{(t,s)/(s,t) \in \varphi\}$ for all φ, and $\mathcal{S}^{\cup}$ be obtained from $\mathcal{S}$ by exercise 19.3. Prove that $\varphi \leq^{\cup} \psi$ iff $\varphi \leq \psi$, $\varphi \circ^{\cup} \psi = \psi\varphi$, $\Pi^{\cup}(\varphi, \psi) = \varphi L \cup \psi R$, $\langle \ \ \rangle^{\cup} = \langle \ \ \rangle$ and $[\ \ ]^{\cup}(\varphi) = \bigcup_n L(\varphi R)^n$.

Hint. Use exercise 5.2 and the equalities $L^{\cup} = L_1$, $R^{\cup} = R_1$.

Another natural bijection in the above example is $\varphi^{\cup} = \neg\varphi = M^2 \backslash \varphi$.

The following exercise shows that the axiom μA_2 does not imply μA_3.

Exercise 19.5. Let $\mathcal{S}$ be the CPS of exercise 16.7 and let the OS $\mathcal{S}'$ be obtained from it by 19.9. Prove that $\mathcal{S}'$ satisfies $(*)_0$ but not μA_3.

Hint. Take $\rho'(\theta)(3s)$, $\rho'(\theta)(3s+2) = \theta(3s)$ and $\rho'(\theta)(3s+1) = \theta(3s+1)$. Then $\rho' \in \mathcal{F}'$ and whenever $\operatorname{Dom} \psi \subset \omega$, then $L'(\rho'((\varphi, \psi))) = \varphi$.

Consider the mapping $\lambda\theta'.\rho'(I', \theta')$ and the regular segment $\mathcal{E}' = \{\theta'/L'\theta' \leq I'\}$. Suppose that $\theta' \in \mathcal{E}'$ and $\theta \in \mathcal{F}$.

If $\operatorname{Dom} \theta'(\theta) \subset \omega$, then $L'(\rho'((\theta, \theta'(\theta)))) = \theta$.

If $\operatorname{Dom} \theta'(\theta) = \omega$, then $L'(\theta'(\theta)) = \omega^2$ and $L'\theta' \leq I'$ implies $L'(\theta'(\theta)) \leq \theta$; hence $\theta = \omega^2$. Therefore, $L'(\rho'((\theta, \theta'(\theta)))) \leq \theta$. We get $L'\rho'(I', \theta')(\theta) \leq \theta$ for all θ, hence $\mathcal{E}'$ is closed under $\lambda\theta'.\rho'(I', \theta')$.

Take $\sigma' = \mu\theta'.\rho'(I', \theta')$. It follows easily that $\operatorname{Dom} \sigma'(I) = \omega$, where $I = \lambda s.s$, hence $L'(\sigma'(I)) = \omega^2 \nleq I = I'(I)$, Therefore, $\sigma' \notin \mathcal{E}'$.

Exercise 19.6. Let $\mathcal{S}'$ be obtained from $\mathcal{S}$ by 19.12. Show that there are $\sigma \in \mathcal{F}$ such that $A_\sigma \notin \mathcal{F}'$.

Hint. Take $\varphi_0 = O$, $\varphi_{n+1} = (I, \varphi_n)$ and $\sigma = [I] = \sup_n \varphi_n$. Show that $\varphi_n < \varphi_{n+1}$; hence $\varphi_n < \sigma$ and $A_\sigma(\varphi_n) = O$ for all n, while $A_\sigma(\sigma) = L$.

Exercise 19.7. Let $\mathcal{S}''$, Tf be the same as in 19.14. Show that they meet the stronger tμ-axiom of exercise 10.9.

Hint. Use exercise 18.7.

Exercise 19.8. Let $\mathcal{S}$, $\mathcal{T}$ satisfy $(***)$, let $\mathcal{S}'$, $\mathcal{T}'$ be obtained from $\mathcal{S}$, $\mathcal{T}$ by 19.12 and let $\mathcal{S}''$, $\mathcal{T}''$ be obtained from $\mathcal{S}'$, $\mathcal{T}'$ again by 19.12. Prove that $\mathcal{S}''$ admits the operation Tf and $\mathcal{S}''$, Tf satisfy tfμA_3 and the stronger axiom of exercise 10.9. Assuming that $\mathcal{S}'''$ is obtained from $\mathcal{S}''$ by 19.12, show that $Tf \in \mathcal{F}'''$.

Hint. Following the proof of 19.14, show in addition that for all φ'', θ' the mapping $Tf(\varphi'')(\theta')$ over $\mathcal{F}$ is $\mathcal{T}$-continuous, while the mapping $Tf(\varphi'')$ over $\mathcal{F}'$ is $\mathcal{T}'$-continuous. (Do not forget that θ' is $\mathcal{T}$-continuous and φ'' is $\mathcal{T}'$-continuous.) Finally, the mapping Tf itself is $\mathcal{T}''$-continuous and preserves types. Use 18.22 and exercise 18.7.

CHAPTER 20

Constructing hierarchies of operative spaces

Given an OS $\mathscr{S}$ and a type set $\mathscr{T}$ satisfying $(*)$ or $(**)$, we construct in this chapter pairs $\mathscr{S}_\xi, \mathscr{T}_\xi$ satisfying the same sufficient condition for all ξ, so that $\{\mathscr{S}_\xi\}$ is a monotonic hierarchy based on $\mathscr{S}$ and $\{\mathscr{S}_\xi\}$ admits transfers. If $\xi = \xi_1 + 1$, then the semigroup $\mathscr{F}_\xi$ consists of all the monotonic and normal mappings over $\mathscr{F}_{\xi_1}$, while if ξ is a limit, then all the preceding spaces $\mathscr{S}_\eta$, $\eta < \xi$ are brought together to form $\mathscr{S}_\xi$. A continuous version of this construction is considered in the exercises.

Proposition 20.1. Let $\mathscr{S}$ be $(*)$-complete ($(**)$-complete). Then a monotonic hierarchy $\{\mathscr{S}_\xi\}$ based on $\mathscr{S}$ can be constructed such that $\mathscr{S}_\xi$ is $(*)$-complete (respectively, $(**)$-complete) for all ξ.

Proof. The proof is by transfinite induction on ξ and closely follows that of 17.2. We first construct a hierarchy of type sets $\{\mathscr{T}_\xi\}$ based on $\mathscr{T}$ by 17.1.

Take $\mathscr{S}_0 = \mathscr{S}$ and $t_0 = t$.

Suppose that $\xi > 0$ and for all $\eta < \xi$ an OS $\mathscr{S}_\eta$ and a type function $t_\eta: \mathscr{F}_\eta \to \mathscr{T}_\eta$ have been constructed such that $\mathscr{S}_\eta, \mathscr{T}_\eta$ satisfy $(*)$ (respectively, $(**)$). If $\eta = \eta_1 + 1 < \xi$, let $\mathscr{S}_\eta$ be obtained from $\mathscr{S}_{\eta_1}$ by 19.11 and identify the subspace $\overline{\mathscr{S}_{\eta_1}}$ of $\mathscr{S}_\eta$ with $\mathscr{S}_{\eta_1}$. If $\eta < \xi$ is a limit ordinal, then let $\mathscr{F}_\eta = \bigcup_{\zeta < \eta} \mathscr{F}_\zeta$ and $\mathscr{S}_\zeta$ be a subspace of $\mathscr{S}_\eta$, whenever $\zeta < \eta$. If $\varphi \in \bigcup_{\eta < \xi} \mathscr{F}_\eta$, then the ordinal $r(\varphi) = \min\{\eta / \varphi \in \mathscr{F}_\eta\}$ is the rank of φ as usual. If $\eta < \xi$ and $\zeta = r(\varphi) < \eta$, then let $t_\zeta(\varphi)$ be represented in $\mathscr{T}_\eta$ by $t_\eta(\varphi)$. This is the inductive hypothesis.

A familiar notation. If $\eta + 1 < \xi$, $r(\varphi) \leq \eta + 1$, $\psi \in \mathscr{F}_\eta$ and φ_1 is the member of $\mathscr{F}_{\eta+1}$ identified with φ, then $\varphi(\psi)_\eta$ will stand for $\varphi_1(\psi)$ as in chapter 15. Observe that $\varphi(\psi)_\eta = \varphi(\psi)$, if $r(\varphi) = \eta + 1$, while $\varphi(\psi)_\eta = \varphi\psi$, if $r(\varphi) < \eta + 1$. Moreover, $\chi \in \mathscr{F}_{\eta+1}$, $\varphi \leqslant \chi$ imply $\varphi(\psi)_\eta \leqslant \chi(\psi)_\eta$.

1. $\xi = \xi_1 + 1$.

Obtain $\mathscr{S}_\xi, t_\xi$ from $\mathscr{S}_{\xi_1}, t_{\xi_1}$ by 19.11 (i.e., by the construction of 19.9) and identify $\overline{\mathscr{S}_{\xi_1}}$ with $\mathscr{S}_{\xi_1}$. This can be done since the IOS $\mathscr{S}_{\xi_1}, \overline{\mathscr{S}_{\xi_1}}$ are isomorphic by 12.24.

If $\varphi \in \mathscr{F}_\xi$ and $\zeta = r(\varphi) < \xi$, then the argument adduced at the corresponding point of the proof of 17.2 implies that $t_\xi(\varphi)$ represents $t_\zeta(\varphi)$ in $\mathscr{T}_\xi$. Therefore, the induction clause holds for ξ.

2. ξ is a limit ordinal.

Take $\mathscr{F}_\xi = \bigcup_{\eta<\xi} \mathscr{F}_\eta$ and $t_\xi(\varphi) = (\zeta, t_\zeta(\varphi))$ for $\varphi \in \mathscr{F}_\xi$, where $\zeta = r(\varphi)$. The proof of 17.1 implies that $t_\xi(\varphi) \in \mathscr{T}_\xi$ and $t_\zeta(\varphi)$ is represented in $\mathscr{T}_\xi$ by $(\zeta, t_\zeta(\varphi))$, i.e. by $t_\xi(\varphi)$.

If $\varphi, \psi \in \mathscr{F}_\xi$, then take $\varphi \leq_\xi \psi$ iff $\varphi \leq_\zeta \psi$, $\varphi \circ_\xi \psi = \varphi \circ_\zeta \psi$ and $(\varphi, \psi)_\xi = (\varphi, \psi)_\zeta$, where $\zeta = \max\{r(\varphi), r(\psi)\}$. It follows from the induction hypothesis that if $\zeta \leq \eta < \xi$, then $\varphi \leq_\xi \psi$ iff $\varphi \leq_\eta \psi$, $\varphi \circ_\xi \psi = \varphi \circ_\eta \psi$ and $(\varphi, \psi)_\xi = (\varphi, \psi)_\eta$, so we may write simply $\varphi \leq \psi, \varphi\psi, (\varphi, \psi)$. The 5-tuple $\mathscr{S}_\xi = (\mathscr{F}_\xi, I, \Pi_\xi, L, R)$ is an OS since $\mathscr{S}_\eta$ is for all $\eta < \xi$.

Our next aim is to verify that $\mathscr{S}_\xi, \mathscr{T}_\xi$ satisfy condition $(*)$ (condition $(**)$).

Suppose that $t_\xi(\varphi) = (\eta, t_\eta(\varphi))$, $t_\xi(\psi) = (\zeta, t_\zeta(\psi))$, $(\xi_1, a) \in \mathscr{T}_\xi$ and $t_\xi(\varphi), t_\xi(\psi) \leq (\xi_1, a)$. Then $\eta, \zeta \leq \xi_1$, and $t_{\xi_1}(\varphi), t_{\xi_1}(\psi)$ represent $t_\eta(\varphi), t_\zeta(\psi)$ in $\mathscr{F}_{\xi_1}$ and $t_{\xi_1}(\varphi)$, $t_{\xi_1}(\psi) \leq a$. It follows that $r(\varphi), r(\psi) \leq \xi_1$, hence $\varphi, \psi \in \mathscr{F}_{\xi_1}$, which implies $\varphi\psi \in \mathscr{F}_{\xi_1}$ and $t_{\xi_1}(\varphi\psi) \leq a$ since $\circ_{\xi_1}$ preserves types. If $\xi_2 = r(\varphi\psi)$, then $\xi_2 \leq \xi_1$, $t_\xi(\varphi\psi) = (\xi_2, t_{\xi_2}(\varphi\psi))$ and $t_{\xi_1}(\varphi\psi)$ represents $t_{\xi_2}(\varphi\psi)$ in $\mathscr{T}_{\xi_1}$; hence $t_\xi(\varphi\psi) \leq (\xi_1, a)$. Therefore, the operation $\circ_\xi$ preserves types. Similarly, Π_ξ preserves types.

Suppose that $a = (\eta, b) \in \mathscr{T}_\xi$ and $\mathscr{H}$ is a well ordered subset of $\mathscr{F}_{\xi,a}$. Then $t_\xi(\theta) \leq a$; hence $\theta \in \mathscr{F}_\eta$ and $t_\eta(\theta) \leq b$ for all $\theta \in \mathscr{H}$. Therefore, $\mathscr{H}$ is a well ordered subset of $\mathscr{F}_{\eta,b}$. It follows from the inductive hypothesis that there is a $\varphi \in \mathscr{F}_{\eta,b}$ such that $\varphi\psi = \sup(\mathscr{H}\psi)$ in $\mathscr{F}_\eta$ for all $\psi \in \mathscr{F}_\eta$. We get $t_\xi(\varphi) \leq a$ from condition (ξ) of 17.1; hence $\varphi \in \mathscr{F}_{\xi,a}$. To complete the proof we shall prove by transfinite induction that $\varphi\psi = \sup(\mathscr{H}\psi)$ in $\mathscr{F}_\zeta$ for all $\psi \in \mathscr{F}_\zeta$, $\eta < \zeta \leq \xi$. In the case of condition $(**)$ the equalities $L\varphi = \sup L\mathscr{H}$, $R\varphi = \sup R\mathscr{H}$ will also be established in $\mathscr{F}_\zeta$.

a. Let $\zeta = \zeta_1 + 1$, $\eta < \zeta < \xi$ and suppose the assertion holds for ζ_1.

It is immediate that $\theta\psi \leq_\zeta \varphi\psi$ for all $\theta \in \mathscr{H}, \psi \in \mathscr{F}_\zeta$. On the other hand, suppose that $\psi, \tau \in \mathscr{F}_\zeta$ and $\theta\psi \leq_\zeta \tau$ for all $\theta \in \mathscr{H}$. If $\sigma \in \mathscr{F}_{\zeta_1}$, then $(\varphi\psi)(\sigma)_{\zeta_1} \leq_{\zeta_1} \tau(\sigma)_{\zeta_1}$, hence $\theta \circ_{\zeta_1} \psi(\sigma)_{\zeta_1} \leq_{\zeta_1} \tau(\sigma)_{\zeta_1}$. The inductive hypothesis implies that $\varphi \circ_{\zeta_1} \psi(\sigma)_{\zeta_1} = \sup(\mathscr{H} \circ_{\zeta_1} \psi(\sigma)_{\zeta_1})$ in $\mathscr{F}_{\zeta_1}$, hence $\varphi \circ_{\zeta_1} \psi(\sigma)_{\zeta_1} \leq_{\zeta_1} \tau(\sigma)_{\zeta_1}$, i.e. $(\varphi\psi)(\sigma)_{\zeta_1} \leq_{\zeta_1} \tau(\sigma)_{\zeta_1}$. This holds for all $\sigma \in \mathscr{F}_{\zeta_1}$, hence $\varphi\psi \leq_\zeta \tau$. Therefore, $\varphi\psi = \sup(\mathscr{H}\psi)$ in $\mathscr{F}_\zeta$.

Suppose that $\mathscr{S}_{\zeta_1}, \mathscr{T}_{\zeta_1}$ satisfy $(**)$. Let $\tau \in \mathscr{T}_\zeta$ and $L\theta \leq_\zeta \tau$ for all $\theta \in \mathscr{H}$. If $\sigma \in \mathscr{F}_{\zeta_1}$, then $(L\theta)(\sigma)_{\zeta_1} \leq_{\zeta_1} \tau(\sigma)_{\zeta_1}$, hence $L\theta\sigma \leq_{\zeta_1} \tau(\sigma)_{\zeta_1}$ for all $\theta \in \mathscr{H}$. The inductive hypothesis implies $L\varphi\sigma = \sup(L\mathscr{H}\sigma)$ in $\mathscr{F}_{\zeta_1}$; hence $L\varphi\sigma \leq_{\zeta_1} \tau(\sigma)_{\zeta_1}$, i.e. $(L\varphi)(\sigma)_{\zeta_1} \leq_{\zeta_1} \tau(\sigma)_{\zeta_1}$. This holds for all $\sigma \in \mathscr{F}_{\zeta_1}$; hence $L\varphi \leq_\zeta \tau$. Therefore, $L\varphi = \sup L\mathscr{H}$ in $\mathscr{F}_\zeta$ and similarly $R\varphi = \sup R\mathscr{H}$ in $\mathscr{F}_\zeta$.

b. Let ζ be a limit ordinal, $\eta < \zeta \leq \xi$ and suppose the assertion holds for all $\eta_1 < \zeta$.

It is immediate that $\theta\psi \leq_\zeta \varphi\psi$ for all $\theta \in \mathscr{H}$, $\psi \in \mathscr{F}_\zeta$. Suppose that $\psi, \tau \in \mathscr{F}_\zeta$ and $\theta\psi \leq_\zeta \tau$ for all $\theta \in \mathscr{H}$. Taking $\eta_1 = \max\{r(\psi), r(\tau), \eta\}$, we get $\eta_1 < \zeta, \psi$, $\tau \in \mathscr{F}_{\eta_1}$, $\mathscr{H} \subseteq \mathscr{F}_{\eta_1}$ and $\theta\psi \leq_{\eta_1} \tau$ for all $\theta \in \mathscr{H}$. The inductive hypothesis gives $\varphi\psi = \sup(\mathscr{H}\psi)$ in $\mathscr{F}_{\eta_1}$, hence $\varphi\psi \leq_{\eta_1} \tau$. It follows that $\varphi\psi \leq_\zeta \tau$; hence $\varphi\psi = \sup(\mathscr{H}\psi)$ in $\mathscr{F}_\zeta$.

Suppose that $\mathscr{S}_{\eta_1}, \mathscr{T}_{\eta_1}$ satisfy $(**)$ for all $\eta_1 < \zeta$. Let $\tau \in \mathscr{F}_\zeta$ and $L\theta \leq_\zeta \tau$ for all $\theta \in \mathscr{H}$. Taking $\eta_1 = \max\{r(\tau), \eta\}$, we get $\eta_1 < \zeta, L, \tau \in \mathscr{F}_{\eta_1}$, $\mathscr{H} \subseteq \mathscr{F}_{\eta_1}$ and

$L\theta \leq_{\eta_1} \tau$ for all $\theta \in \mathscr{H}$. The inductive hypothesis gives $L\varphi = \sup L\mathscr{H}$ in $\mathscr{F}_{\eta_1}$; hence $L\varphi \leq_{\eta_1} \tau$ which implies $L\varphi \leq_{\zeta} \tau$. Therefore, $L\varphi = \sup L\mathscr{H}$ in $\mathscr{F}_{\zeta}$ and similarly $R\varphi = \sup R\mathscr{H}$ in $\mathscr{F}_{\zeta}$. We conclude that $\mathscr{S}_{\xi}$, $\mathscr{T}_{\xi}$ satisfy $(*)$ (respectively, $(**)$), while it is immediate that $\mathscr{S}_{\eta}$ is a subspace of $\mathscr{S}_{\xi}$, provided $\eta < \xi$. The proof is complete.

Proposition 20.2. Let $\mathscr{S}$ be $(*)$-complete ($(**)$-complete) and let $\{\mathscr{S}_{\xi}\}$ be the monotonic hierarchy based on $\mathscr{S}$ constructed in 20.1. Then $\{\mathscr{S}_{\xi}\}$ admits transfers, i.e. for all ξ the element $Tf_{\xi+2} \in \mathscr{F}_{\xi+3}$ is a transfer operation over $\mathscr{F}_{\xi+2}$ satisfying the axiom $\mathrm{tf}\mu\mathrm{A}_2$ (respectively, $\mathrm{tf}\mu\mathrm{A}_3$). The stronger axiom of exercise 10.9 is also valid.

This follows from 19.14 and exercise 19.7.

Proposition 20.3. Let $\{\mathscr{S}_{\xi}\}$ be the hierarchy constructed by 20.1. Then for all ξ there are members $Q_{\sigma,\xi}$ (and $Q_{\sigma,\mathscr{C},\xi}$, provided L, R have an upper bound U) of $\mathscr{F}_{\xi+1}$ which are the mappings Q_{σ} (and $Q_{\sigma,\mathscr{C}}$) corresponding to $\mathscr{S}_{\xi}$.

This follows from 19.13.

Suppose that $\mathscr{S}$ is $(*)_0$-complete or $(**)_0$-complete and a hierarchy $\{\mathscr{S}_{\xi}\}$ based on $\mathscr{S}$ is constructed by 20.1. Then all $\mathscr{S}_n, n < \omega$ satisfy the same sufficient condition and types arise beyond ω, exactly as they did in hierarchies of pairing spaces. (Cf. exercises 17.1–17.3.)

EXERCISES TO CHAPTER 20

Exercise 20.1. Let $\mathscr{S}$ be a $(***)$-complete OS. Construct a monotonic hierarchy $\{\mathscr{S}_{\xi}\}$ based on $\mathscr{S}$.

Hint. Follow the proof of 20.1, substituting $(***)$ for $(*)$ and 19.12 for 19.11 in the induction clause.

Remarks. Hierarchies constructed this way are more appropriately called *continuous* since for all ξ the semigroup $\mathscr{F}_{\xi+1}$ consists of $\mathscr{T}_{\xi}$-continuous mappings over $\mathscr{F}_{\xi}$. Exercise 19.8 ensures that all continuous hierarchies admit transfers, while exercise 19.6 shows that proposition 20.3 does not hold for continuous hierarchies.