PART C

Diversified recursion theory on operative spaces

CHAPTER 10

Translation-like operations

An IOS $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ has four initial operations, namely $\circ$, Π, $\langle\ \ \rangle$, $[\ \ ]$. Since our intention when choosing them in chapter 2 was to get a minimal collection, it may happen in some spaces that there exist wider collections of initial operations which also support a sensible recursion theory.

On the other hand, the role played so far by the operation $\langle\ \ \rangle$ suggests that similar operations are 'nice' from the recursion theoretic viewpoint. Roughly speaking, such a translation-like operation $\langle\!\langle\ \ \rangle\!\rangle$ maps $\mathscr{F}$ injectively into itself, transforming $\circ$, Π, $\langle\ \ \rangle$, $[\ \ ]$, $\langle\!\langle\ \ \rangle\!\rangle$ into $\langle\!\langle\ \ \rangle\!\rangle$-free operations, e.g. $\langle\!\langle \varphi\psi \rangle\!\rangle = \Gamma(\langle\!\langle \varphi \rangle\!\rangle, \langle\!\langle \psi \rangle\!\rangle)$ for a certain mapping Γ recursive in $\mathscr{F}$.

The present chapter shows that a good deal of the theory developed in chapter 9 can actually be reestablished in this more complicated situation. Of particular interest is the axiomatically defined operation St. An example of a translation-like operation is given in the exercises below, while others are considered in chapters 21, 22, 24, 25, 28, 30. The concept of translation-like operation is important for the considerations in chapters 14, 15 and crucial for the study of Skordev combinatory spaces in chapter 27.

Formally, assume that a monotonic operation $\langle\!\langle\ \ \rangle\!\rangle : \mathscr{F} \to \mathscr{F}$ and fixed elements $K_0, \ldots, K_m \in \mathscr{F}$ are given together with mappings Σ_0, $\Sigma_1 : \mathscr{F}^2 \to \mathscr{F}$, $\Sigma_2 - \Sigma_5 : \mathscr{F} \to \mathscr{F}$ recursive in $\mathscr{B}_0 = \{K_0, \ldots, K_m, \langle\!\langle K_0 \rangle\!\rangle, \ldots, \langle\!\langle K_m \rangle\!\rangle\}$ such that

(0) $\langle\!\langle \varphi\psi \rangle\!\rangle = \Sigma_0(\langle\!\langle \varphi \rangle\!\rangle, \langle\!\langle \psi \rangle\!\rangle)$,

(1) $\langle\!\langle (\varphi, \psi) \rangle\!\rangle = \Sigma_1(\langle\!\langle \varphi \rangle\!\rangle, \langle\!\langle \psi \rangle\!\rangle)$,

(2) $\langle\!\langle \langle \varphi \rangle \rangle\!\rangle = \Sigma_2(\langle\!\langle \varphi \rangle\!\rangle)$,

(3) $\langle\!\langle [\varphi] \rangle\!\rangle = \Sigma_3(\langle\!\langle \varphi \rangle\!\rangle)$,

(4) $\langle\!\langle \langle\!\langle \varphi \rangle\!\rangle \rangle\!\rangle = \Sigma_4(\langle\!\langle \varphi \rangle\!\rangle)$,

(5) $\varphi = \Sigma_5(\langle\!\langle \varphi \rangle\!\rangle)$

for all φ, ψ. Under these conditions $\langle\!\langle\ \ \rangle\!\rangle$ is said to be a t-*operation*. It is convenient to include I, L, R in the list $K_0, \ldots, K_m$. On the other hand, members of $\mathscr{B}_0$ will sometimes be dropped, provided they can be obtained from remaining ones by means of the basic IOS-operations.

The operation $\langle\ \ \rangle$ is itself a t-operation with mappings $\Sigma_0 = \lambda\theta_1\theta_2.\theta_1\theta_2$, $\Sigma_1 = \lambda\theta_1\theta_2.C(\theta_1, \theta_2)$, $\Sigma_2 = \Sigma_4 = \lambda\theta.P\theta Q$, $\Sigma_3 = \lambda\theta.C[\theta C]$, $\Sigma_5 = \lambda\theta.L\theta[L]$ which are prime recursive in C, P, Q.

It would appear that the basic features of t-operations are (0), (1), (4).

Actually, whenever $\langle\!\langle\ \rangle\!\rangle$ satisfies (0)–(4), then it is easily seen that $\langle\!\langle\ \rangle\!\rangle_1 = \lambda\theta.(\theta, \langle\!\langle\theta\rangle\!\rangle)$ satisfies (0)–(5), while it will be shown below that (0), (1) imply (2), (3) by means of a modified μ-axiom.

An element φ is t-*recursive in* $\mathscr{B} \subseteq \mathscr{F}$ if

$$\varphi \in cl(\mathscr{B}_0 \cup \mathscr{B}/\circ, \Pi, \langle\ \ \rangle, [\ \ ], \langle\!\langle\ \ \rangle\!\rangle)$$

and similarly for mappings. The notions of prime recursiveness etc. are also extended in this way.

The following Parametrized Pull Back Theorem plays a key role here by ensuring that all t-recursive mappings have the property given in (0)–(4) for the initial operations $\circ$, Π, $\langle\ \ \rangle$, $[\ \ ]$, $\langle\!\langle\ \ \rangle\!\rangle$.

Proposition 10.1. If Γ is a unary mapping t-recursive in $\mathscr{B}$, then there is a mapping Γ^* recursive in $\mathscr{B}_0 \cup \langle\!\langle\mathscr{B}\rangle\!\rangle$ such that $\Gamma = \lambda\theta.\Gamma^*(\langle\!\langle\theta\rangle\!\rangle)$.

Proof. It suffices to construct a mapping Γ^{**} recursive in $\mathscr{B}_0 \cup \langle\!\langle\mathscr{B}\rangle\!\rangle$ such that $\langle\!\langle\Gamma(\theta)\rangle\!\rangle = \Gamma^{**}(\langle\!\langle\theta\rangle\!\rangle)$ for all θ and then take $\Gamma^* = \lambda\theta.\Sigma_5(\Gamma^{**}(\theta))$.

Take $(\lambda\theta.\theta)^{**} = \lambda\theta.\theta$, $(\lambda\theta.\psi)^{**} = \lambda\theta.\langle\!\langle\psi\rangle\!\rangle$ for all $\psi \in \{K_0, \ldots, K_m\} \cup \mathscr{B}$, $(\lambda\theta.\langle\!\langle\psi\rangle\!\rangle)^{**} = \lambda\theta.\Sigma_4(\langle\!\langle\psi\rangle\!\rangle)$ for all $\psi \in \{K_0, \ldots, K_m\}$.

Let Γ_1^{**}, Γ_2^{**} correspond respectively to Γ_1, Γ_2. Then take

$$(\lambda\theta.\Gamma_1(\theta)\Gamma_2(\theta))^{**} = \lambda\theta.\Sigma_0(\Gamma_1^{**}(\theta), \Gamma_2^{**}(\theta)),$$
$$(\lambda\theta.(\Gamma_1(\theta), \Gamma_2(\theta)))^{**} = \lambda\theta.\Sigma_1(\Gamma_1^{**}(\theta), \Gamma_2^{**}(\theta)),$$
$$(\lambda\theta.\langle\Gamma_1(\theta)\rangle)^{**} = \lambda\theta.\Sigma_2(\Gamma_1^{**}(\theta)),$$
$$(\lambda\theta.[\Gamma_1(\theta)])^{**} = \lambda\theta.\Sigma_3(\Gamma_1^{**}(\theta)),$$
$$(\lambda\theta.\langle\!\langle\Gamma_1(\theta)\rangle\!\rangle)^{**} = \lambda\theta.\Sigma_4(\Gamma_1^{**}(\theta)).$$

This completes the proof.

As an easy corollary we get a Pull Back Theorem for elements.

Proposition 10.2. An element φ is t-recursive in $\mathscr{B}$ iff it is recursive in $\mathscr{B}_0 \cup \langle\!\langle\mathscr{B}\rangle\!\rangle$. In particular, φ is t-recursive iff it is recursive in $\mathscr{B}_0$.

Proof. If φ is t-recursive in $\mathscr{B}$, then so is the mapping $\Gamma = \lambda\theta.\varphi$. Take Γ^* to correspond to Γ by 10.1. Then $\varphi = \Gamma^*(\langle\!\langle I\rangle\!\rangle)$; hence φ is recursive in $\mathscr{B}_0 \cup \langle\!\langle\mathscr{B}\rangle\!\rangle$, which completes the proof.

Relative t-recursiveness possesses the ordinary properties stated at the beginning of chapter 7. Certain specific characterizations may be obtained by 7.11, 10.2. Analogs to 7.18–7.20 for mappings t-recursive in $\mathscr{B}$ may also be established.

The Pull Back Theorems and the normal form results 9.3–9.7 imply corresponding Normal t-Form Theorems.

Proposition 10.3. If φ is t-recursive in $\mathscr{B}$, then $\varphi = \bar{1}[\sigma]$ with a certain σ strictly primitive in $\mathscr{B}_0 \cup \langle\!\langle\mathscr{B}\rangle\!\rangle$. This follows from 10.2, 9.3.

Proposition 10.4. If Γ is a n-ary mapping t-recursive in $\mathscr{B}$, then

$$\Gamma = \lambda\theta_1 \ldots \theta_n.\bar{1}[\varphi(I, \langle\langle\!\langle\theta_1\rangle\!\rangle\rangle, \ldots, \langle\langle\!\langle\theta_n\rangle\!\rangle\rangle)]$$

with a certain φ strictly primitive in $\mathscr{B}_0 \cup \langle\!\langle\mathscr{B}\rangle\!\rangle$.

Proof. Proposition 10.1 and an analogue to 7.20 imply that there is an n-ary mapping Γ^* recursive in $\mathscr{B}_0 \cup \langle \mathscr{B} \rangle$ such that $\Gamma = \lambda\theta_1 \ldots \theta_n.\ \Gamma^*(\langle \theta_1 \rangle, \ldots, \langle \theta_n \rangle)$. Applying 9.4 to Γ^* we get the desired normal form.

Two t-Enumeration Theorems follow.

Proposition 10.5. If $\mathscr{B}$ is finite, then there is a unary mapping Σ t-recursive in $\mathscr{B}$ and universal for the unary mappings t-recursive in $\mathscr{B}$.

Proof. Take a unary mapping Σ^* recursive in $\mathscr{B}_0 \cup \langle \mathscr{B} \rangle$ and universal for the unary mappings recursive in $\mathscr{B}_0 \cup \langle \mathscr{B} \rangle$ by 9.19. Then, using 10.1 we see that $\Sigma = \lambda\theta.\Sigma^*(\langle \theta \rangle)$ is universal for the unary mappings t-recursive in $\mathscr{B}$.

Proposition 10.6. If $\mathscr{B}$ is finite, then there is an element σ t-recursive in $\mathscr{B}$ and universal for the elements t-recursive in $\mathscr{B}$.

Proof. Take the mapping Σ of the previous statement. The element $\sigma = \Sigma(I)$ is t-recursive in $\mathscr{B}$ and whenever φ is t-recursive in $\mathscr{B}$, then so is $\Gamma = \lambda\theta.\varphi$; hence there is n such that $\Gamma = \lambda\theta.\bar{n}\Sigma(\theta)$, which implies $\varphi = \bar{n}\sigma$. This completes the proof.

Let $\mathscr{B}$ be a fixed subset of $\mathscr{F}$, let $\mathscr{U}$ stand for the set of all elements t-recursive in $\mathscr{B}$ and $\mathscr{M}$ for the set of all unary mappings t-recursive in $\mathscr{B}$. Notions of element and mapping principal universal respectively for $\mathscr{U}$, $\mathscr{M}$ are introduced by the same definitions as in chapter 9. All the further statements and proofs of that chapter then remain valid in the present context. However, the analog to the Recursion Theorem 9.23 requires the following t-Transition Theorem.

Proposition 10.7. Let Γ be a unary mapping t-recursive in $\mathscr{B}$. Then there is a mapping Γ^* t-recursive in $\mathscr{B}$ such that $\Gamma^*(\theta) = (\Gamma(L\theta), \Gamma^*(R\theta))$ for all θ. In particular, $\bar{n}\Gamma^*(\theta) = \Gamma(\bar{n}\theta)$ for all n, θ.

Proof. It suffices to consider the mapping $\langle\ \rangle$. Let a t-recursive mapping $\langle\ \rangle^*$ correspond to $\langle\ \rangle$ such that $\langle \theta \rangle^* = (\langle L\theta \rangle, \langle R\theta \rangle^*)$ for all θ. Then, given a unary mapping Γ t-recursive in $\mathscr{B}$, take Γ_1 to correspond to Γ by 10.1, Γ_1^* to correspond to Γ_1 by 7.21 and finally take $\Gamma^* = \lambda\theta.\Gamma_1^*(\langle \theta \rangle^*)$. It follows that

$$\begin{aligned}\Gamma^*(\theta) &= (\Gamma_1(L\langle \theta \rangle^*), \Gamma_1^*(R\langle \theta \rangle^*))\\ &= (\Gamma_1(\langle L\theta \rangle), \Gamma_1^*(\langle R\theta \rangle^*)) = (\Gamma(L\theta), \Gamma^*(R\theta))\end{aligned}$$

for all θ; hence Γ^* corresponds to Γ.

The mappings $\Gamma_1 = \lambda\theta.\Sigma_0(\langle L \rangle, \theta)$, $\Gamma_2 = \lambda\theta.\Sigma_0(\langle R \rangle, \theta)$ are recursive in $\mathscr{B}_0$. Take a mapping Γ_1^* to correspond to Γ_1 by 7.21 and a mapping Γ_3 recursive in $\mathscr{B}_0$ such that $\Gamma_3(\theta) = (\theta, \Gamma_3(\Gamma_2(\theta)))$ for all θ by exercise 9.8, then take $\langle\ \rangle^* = \lambda\theta.\Gamma_1^*(\Gamma_3(\langle \theta \rangle))$. It follows that

$$\begin{aligned}\langle \theta \rangle^* &= \Gamma_1^*(\Gamma_3(\langle \theta \rangle)) = (\Gamma_1(L\Gamma_3(\langle \theta \rangle)), \Gamma_1^*(R\Gamma_3(\langle \theta \rangle)))\\ &= (\Gamma_1(\langle \theta \rangle), \Gamma_1^*(\Gamma_3(\Gamma_2(\langle \theta \rangle))))\\ &= (\langle L\theta \rangle, \Gamma_1^*(\Gamma_3(\langle R\theta \rangle)))\\ &= (\langle L\theta \rangle, \langle R\theta \rangle^*)\end{aligned}$$

for all θ, which completes the proof.

A Generalized t-Recursion Theorem may also be established by adapting the hint to exercise 9.8. To obtain a First t-Recursion Theorem, however, one has to employ a stronger μ-axiom.

The notion of t-*inductive mapping* is introduced by modifying the definition given in chapter 5. Namely, replace $\{I, L, R\}$ by $\mathscr{B}_0$ in the first clause and add to the second clause the following case of $\langle\!\langle\ \ \rangle\!\rangle$:

If $\Gamma_1 : \mathscr{F}^n \to \mathscr{F}$ is t-inductive, then so is $\Gamma = \lambda\theta_1 \ldots \theta_n . \langle\!\langle \Gamma_1(\theta_1, \ldots, \theta_n) \rangle\!\rangle$.

A t-*simple segment* is a subset of $\mathscr{F}$ of the form $\{\theta / \langle\!\langle \theta \rangle\!\rangle \leq \Sigma_6(\tau)\}$, where Σ_6 is a given mapping recursive in $\mathscr{B}_0$ such that $\Sigma_6(\langle\!\langle \theta \rangle\!\rangle) = \langle\!\langle \theta \rangle\!\rangle$ for all θ.

Axiom tμA. For any $n+1$-ary t-inductive mapping Γ and any $\theta_1, \ldots, \theta_n$ the inequality $\Gamma(\theta_1, \ldots, \theta_n, \theta) \leq \theta$ has a solution which is a member of all t-simple segments closed under the mapping $\lambda\theta . \Gamma(\theta_1, \ldots, \theta_n, \theta)$.

The axioms μA_1, μA_2, μA_3 are now completed by axiom tμA,

$$t\mu A_i = \mu A_i + t\mu A, \quad i = 1, 2, 3.$$

The use of a modified μ-axiom will be indicated by the corresponding number of asterisks.

We now prove the First t-Recursion Theorem.

Proposition 10.8*. If Γ is a unary mapping t-recursive in $\mathscr{B}$, then the element $\mu\theta . \Gamma(\theta)$ exists and is t-recursive in $\mathscr{B}$.

Proof. We modify the translation method of the proof of 9.12* so that the role of $\langle\ \ \rangle$ is played by $\langle\!\langle\ \ \rangle\!\rangle$.

Let θ_0 be the solution to $\Gamma(\theta) \leq \theta$ assumed by tμA, let Γ^* correspond to $\lambda\theta . \langle\!\langle \Gamma(\theta) \rangle\!\rangle$ by 10.1 and let $\Gamma_1 = \lambda\theta . \Sigma_6(\Gamma^*(\theta))$. Then Γ_1 is recursive in $\mathscr{B}_0 \cup \langle\!\langle \mathscr{B} \rangle\!\rangle$, hence $\theta_1 = \mu\theta . \Gamma_1(\theta)$ exists and is recursive in $\mathscr{B}_0 \cup \langle\!\langle \mathscr{B} \rangle\!\rangle$ by 9.13*.

It follows that

$$\langle\!\langle \Gamma(\theta) \rangle\!\rangle = \Sigma_6(\langle\!\langle \Gamma(\theta) \rangle\!\rangle) = \Sigma_6(\Gamma^*(\langle\!\langle \theta \rangle\!\rangle)) = \Gamma_1(\langle\!\langle \theta \rangle\!\rangle)$$

for all θ. In particular, $\Gamma_1(\langle\!\langle \theta_0 \rangle\!\rangle) = \langle\!\langle \Gamma(\theta_0) \rangle\!\rangle \leq \langle\!\langle \theta_0 \rangle\!\rangle$ implies $\theta_1 \leq \langle\!\langle \theta_0 \rangle\!\rangle$. On the other hand, consider the t-simple segment

$$\mathscr{E} = \{\theta / \langle\!\langle \theta \rangle\!\rangle \leq \theta_1\} = \{\theta / \langle\!\langle \theta \rangle\!\rangle \leq \Sigma_6(\Gamma^*(\theta_1))\}.$$

If $\theta \in \mathscr{E}$, then

$$\langle\!\langle \Gamma(\theta) \rangle\!\rangle = \Gamma_1(\langle\!\langle \theta \rangle\!\rangle) \leq \Gamma_1(\theta_1) = \theta_1,$$

hence $\langle\!\langle \theta_0 \rangle\!\rangle \in \mathscr{E}$ by tμA. Therefore, $\langle\!\langle \theta_0 \rangle\!\rangle = \theta_1$, hence $\theta_0 = \Sigma_5(\theta_1)$ and θ_0 is t-recursive in $\mathscr{B}$.

We still have to show that $\theta_0 = \mu\theta . \Gamma(\theta)$. If $\Gamma(\tau) \leq \tau$, we get $\Gamma_1(\langle\!\langle \tau \rangle\!\rangle) \leq \langle\!\langle \tau \rangle\!\rangle$, hence $\theta_1 \leq \langle\!\langle \tau \rangle\!\rangle$ which implies $\theta_0 = \Sigma_5(\theta_1) \leq \Sigma_5(\langle\!\langle \tau \rangle\!\rangle) = \tau$. This completes the proof.

Analogues to 9.14*–9.16* and exercises 9.3*, 9.4 may also be established for t-recursiveness. In particular, it follows from the analogue to 9.15* and the proof of 10.8* that the solution to the inequality $\Gamma(\theta_1, \ldots, \theta_n, \theta) \leq \theta$ assumed in tμA is exactly $\mu\theta . \Gamma(\theta_1, \ldots, \theta_n, \theta)$. (This fact will be used in the proofs of 10.9*, 10.10*.)

The following two statements show that the existence of mappings Σ_0, Σ_1 implies the existence of Σ_2, Σ_3 by $t\mu A_1$.

Proposition 10.9*. Let Σ_0, Σ_1 satisfy (0), (1). Then there is a unary mapping Σ_2 recursive in $\mathscr{B}_0$ satisfying (2).

Proof. Take

$$\Gamma = \lambda\theta\theta_1.\Sigma_6(\Sigma_1(\Sigma_0(\theta, \langle\!\langle L\rangle\!\rangle), \Sigma_0(\theta_1, \langle\!\langle R\rangle\!\rangle)))$$

and $\Sigma_2 = \lambda\theta.\mu\theta_1.\Gamma(\theta, \theta_1)$. The mapping Γ is recursive in $\mathscr{B}_0$, hence so is Σ_2 by 9.14*. It follows that

$$\Gamma(\langle\!\langle\varphi\rangle\!\rangle, \langle\!\langle\langle\varphi\rangle\rangle\!\rangle) = \Sigma_6(\langle\!\langle(\varphi L, \langle\varphi\rangle R)\rangle\!\rangle) = \Sigma_6(\langle\!\langle\langle\varphi\rangle\rangle\!\rangle) = \langle\!\langle\langle\varphi\rangle\rangle\!\rangle,$$

hence $\Sigma_2(\langle\!\langle\varphi\rangle\!\rangle) \le \langle\!\langle\langle\varphi\rangle\rangle\!\rangle$. On the other hand, consider the t-simple segment $\mathscr{E} = \{\theta/\langle\!\langle\theta\rangle\!\rangle \le \Sigma_2(\langle\!\langle\varphi\rangle\!\rangle)\}$. If $\theta\in\mathscr{E}$, then

$$\langle\!\langle(\varphi L, \theta R)\rangle\!\rangle = \Gamma(\langle\!\langle\varphi\rangle\!\rangle, \langle\!\langle\theta\rangle\!\rangle) \le \Gamma(\langle\!\langle\varphi\rangle\!\rangle, \Sigma_2(\langle\!\langle\varphi\rangle\!\rangle)) = \Sigma_2(\langle\!\langle\varphi\rangle\!\rangle),$$

hence $\langle\!\langle\langle\varphi\rangle\rangle\!\rangle \le \Sigma_2(\langle\!\langle\varphi\rangle\!\rangle)$ by $t\mu A$. This completed the proof.

Proposition 10.10*. Let Σ_0, Σ_1 satisfy (0), (1). Then there is a mapping Σ_3 recursive in $\mathscr{B}_0$ satisfying (3).

The proof repeats that of 10.9* with

$$\Gamma = \lambda\theta\theta_1.\Sigma_6(\Sigma_1(\langle\!\langle I\rangle\!\rangle, \Sigma_0(\theta, \theta_1)))$$

and $\Sigma_3 = \lambda\theta.\mu\theta_1.\Gamma(\theta, \theta_1)$.

We bring this chapter to a close by introducing a specific t-operation which subsumes almost all the t-operations studied in the book.

Let $\mathscr{L}$ be a nonempty subset of $\mathscr{F}$, $St:\mathscr{F}\to\mathscr{F}$ (*storing operation*), K_0, K_1, $K_2\in\mathscr{F}$ and suppose that the following axioms be satisfied with x, y ranging over $\mathscr{L}$.

$$\begin{aligned}
&sA_1.\ xK_0(L, R) = (xL, xR).\\
&sA_2.\ xyK_1\in\mathscr{L}, \quad xyK_1K_2 = xy.\\
&(\$)\ \ xSt(\varphi) = \varphi x,\\
&\qquad \forall x(\varphi x\psi \le \rho x\sigma) \Rightarrow St(\varphi)\psi \le St(\rho)\sigma.
\end{aligned}$$

One may assume without loss of generality that $K_0 = St(I)K_0$. For otherwise $St(I)K_0$ satisfies sA_1 and $St(I)K_0 = St(I)St(I)K_0$ by (\$), hence K_0 can be replaced by $St(I)K_0$. Similarly, we assume $K_1 = St(St(I))K_1$.

Propositions 6.35, 5.6, 5.7** and exercise 6.6 imply that $\langle\ \ \rangle$ is a storing operation with $\mathscr{L} = \{\bar{n}/n\in\omega\}$, $K_0 = C$, $K_1 = P$ and $K_2 = Q$.

We now establish some properties of the storing operation.

Proposition 10.11. St is monotonic.

Proof. If $\varphi \le \psi$, then $\varphi x \le \psi x$ for all x; hence $St(\varphi) \le St(\psi)$ by (\$).

Proposition 10.12. $St(\varphi\psi) = St(\varphi)St(\psi)$.

Proof. We have $\psi x = xSt(\psi)$; hence $\varphi\psi x = \varphi xSt(\psi)$ for all x, which implies $St(\varphi\psi) = St(\varphi)St(\psi)$ by (\$).

Proposition 10.13. $St((\varphi, \psi)) = K_0(St(\varphi), St(\psi))$.

Proof. The equalities

$$(\varphi, \psi)x = (\varphi x, \psi x) = (xSt(\varphi), xSt(\psi)) = xK_0(St(\varphi), St(\psi))$$

imply $St((\varphi, \psi)) = K_0(St(\varphi), St(\psi))$ by (\$).

Proposition 10.14. There is a mapping Σ_2 recursive in $K_0, St(L), St(R)$ such that $St(\langle \varphi \rangle) = \Sigma_2(St(\varphi))$ for all φ.

Proof. The mapping

$$\Sigma_2 = \lambda\theta . \mu\theta_1 . K_0(\theta St(L), \theta_1 St(R))$$

exists and is recursive in K_0, $St(L)$, $St(R)$ by 6.38, 6.39. It follows that

$$K_0(St(\varphi)St(L), St(\langle \varphi \rangle)St(R)) = St((\varphi L, \langle \varphi \rangle R)) = St(\langle \varphi \rangle),$$

hence $\Sigma_2(St(\varphi)) \leq St(\langle \varphi \rangle)$. On the other hand, $Rx = xSt(R)$ and

$$(\varphi Lx, x\Sigma_2(St(\varphi))St(R)) = xK_0(St(\varphi)St(L), \Sigma_2(St(\varphi))St(R)) = x\Sigma_2(St(\varphi))$$

imply $\langle \varphi \rangle x \leq x\Sigma_2(St(\varphi))$ for all x by (£); hence

$$St(\langle \varphi \rangle) \leq St(I)\Sigma_2(St(\varphi)) = \Sigma_2(St(\varphi))$$

by (\$). The proof is complete.

Proposition 10.15. $St([\varphi]) = K_0[St(\varphi)K_0]St(I)$.

Proof. We have

$$K_0(St(I), St(\varphi)St([\varphi])) = St((I, \varphi[\varphi])) = St([\varphi]),$$

hence $K_0[St(\varphi)K_0]St(I) \leq St([\varphi])$ by 6.10. On the other hand,

$$\begin{aligned}(x, \varphi xK_0[St(\varphi)K_0]St(I)) &= (xSt(I), xSt(\varphi)K_0[St(\varphi)K_0]St(I)) \\ &= xK_0(I, St(\varphi)K_0[St(\varphi)K_0])St(I) \\ &= xK_0[St(\varphi)K_0]St(I);\end{aligned}$$

hence $[\varphi]x \leq xK_0[St(\varphi)K_0]St(I)$ for all x by (££). Therefore, $St([\varphi]) \leq K_0[St(\varphi)K_0]St(I)$ by (\$), which completes the proof.

Proposition 10.16. $St(St(\varphi)) = K_1St(\varphi)K_2$.

Proof. We have

$$\varphi xy = \varphi xyK_1K_2 = xyK_1St(\varphi)K_2$$

for all x, y; hence $St(\varphi)y = St(I)yK_1St(\varphi)K_2$ for all y, which implies

$$St(St(\varphi)) = St(St(I))K_1St(\varphi)K_2 = K_1St(\varphi)K_2.$$

Proposition 10.17 (Storing Operation Theorem). The operation $\langle\!\langle\ \rangle\!\rangle = \lambda\varphi.(\varphi, St(\varphi))$ is a t-operation with functional elements K_0, K_1, K_2. (So the corresponding set $\mathscr{B}_0$ consists of L, R, K_0, K_1, K_2, $St(I)$, $St(L)$, $St(R)$, $St(K_0)$, $St(K_1)$, $St(K_2)$.)

This follows from 10.12–10.16.

Proposition 10.18. Let $I \in \mathscr{L} \circ \mathscr{F}$. Then St is a t-operation and taking $\Sigma_6 = \lambda\theta . St(I)\theta$, all t-simple segments are normal.

Proof. Let $I = K_3 K_4$ for certain $K_3 \in \mathscr{L}$, $K_4 \in \mathscr{F}$. Then $\varphi = \varphi K_3 K_4 = K_3 St(\varphi) K_4$ for all φ, hence St is a t-operation with functional elements $K_0 - K_4$ by 10.12–10.16. Every t-simple segment $\mathscr{E} = \{\theta / St(\theta) \leq St(I)\tau\}$ is normal since $\mathscr{E} = \{\theta / \forall x(\theta x \leq x\tau)\}$ by ($). The proof is complete.

Proposition 10.19.** Let $\psi_0, \psi_1 \in \mathscr{F}$ such that $\bar{n}\psi_0 \in \mathscr{L}$ and $\bar{n}\psi_0\psi_1 = \bar{n}$ for all n. Then the operation $\langle\ \ \rangle$ can be expressed in terms of St and the elements ψ_0, ψ_1. (Assuming without loss of generality that $\psi_0 = \langle I \rangle \psi_0$.)

Since $\langle\ \ \rangle$ is a storing operation, 10.19** is a particular instance of the following more general statement.

Proposition 10.20. Let St, St^* be storing operations with corresponding sets $\mathscr{L}$, $\mathscr{L}^*$ and let $\psi_0, \psi_1 \in \mathscr{F}$ such that $\mathscr{L}\psi_0 \subseteq \mathscr{L}^*$, $\psi_0\psi_1 = St(I)$. Then St can be expressed in terms of St^* and ψ_0, ψ_1. If $\mathscr{L}\psi_0 = \mathscr{L}^*$ also holds, then St^* can also be expressed in terms of St, ψ_0 and ψ_1. (Assuming without loss of generality that $\psi_0 = St(I)\psi_0$, $\psi_1 = St^*(I)\psi_1$.)

Proof. We have

$$\varphi x = \varphi x \psi_0 \psi_1 = x\psi_0 St^*(\varphi)\psi_1$$

for all $x \in \mathscr{L}$, hence

$$St(\varphi) = St(I)\psi_0 St^*(\varphi)\psi_1 = \psi_0 St^*(\varphi)\psi_1$$

by ($). If $\mathscr{L}\psi_0 = \mathscr{L}^*$, then for all $x^* \in \mathscr{L}^*$ there is a $x \in \mathscr{L}$ such that $x^* = x\psi_0$, hence

$$\varphi x^* = \varphi x \psi_0 = x St(\varphi)\psi_0 = x\psi_0\psi_1 St(\varphi)\psi_0 = x^*\psi_1 St(\varphi)\psi_0$$

which implies $St^*(\varphi) = \psi_1 St(\varphi)\psi_0$ by ($). This completes the proof.

EXERCISES TO CHAPTER 10

Exercise 10.1. Let $\mathscr{S}$ be the IOS of example 4.7 or 4.8 and $J: M^2 \to M$ be injective. Define $St(\varphi)$ by $St(\varphi)(J(s,t)) = J(s, \varphi(t))$ and $St(\varphi)(s)\uparrow$ otherwise. Prove that St is a t-operation and $\mathscr{S}$, St satisfy the axiom $t\mu A_3$.

Hint. Take $\mathscr{L} = \{\tilde{s}/s \in M\}$, where $\tilde{s} = \lambda t . J(s,t)$. Specify elements K_0, K_1, K_2 to satisfy sA_1, sA_2. Use exercise 5.3 and 10.18.

In spaces with t-operations the operation $\langle\ \ \rangle$ is alsmost always expressible in terms of $⟬\ \ ⟭$, i.e. $\langle\ \ \rangle$ is prime t-recursive in certain elements which can be added to $\mathscr{B}_0$. A useful criterion for this is provided by the proof of 5.13: Whenever the axiom μA_3 holds for $\lambda\theta_1\theta . (I, \theta_1\theta)$ and there is a prime t-recursive mapping $\langle\ \ \rangle_1$ such that $\bar{n}\langle\varphi\rangle_1 = \varphi\bar{n}$ for all n, φ, then the operation $\langle\ \ \rangle$ is prime t-recursive. Let us, accordingly, discuss this situation.

In the next four exercises the operation $\langle\ \ \rangle$ is assumed to be prime t-recursive and the mappings $\Sigma_0, \Sigma_1, \Sigma_3 - \Sigma_5$ to be prime recursive in $\mathscr{B}_0$. (Σ_2 is no longer necessary.) It follows immediately that relative t-recursiveness and relative prime t-recursiveness are equivalent, both for elements and mappings.

Exercise 10.2. Prove that 10.1, 10.2 take place with 'prime recursive' substituted for 'recursive'. In particular, there is a mapping Σ_7 prime recursive in $\mathscr{B}_0$ such that $\langle\varphi\rangle = \Sigma_7(\langle\!\langle\varphi\rangle\!\rangle)$ for all φ.

The normal Form Theorems 10.3, 10.4 can be further specified.

Exercise 10.3. Prove that whenever φ is t-recursive in $\mathscr{B}$, then $\varphi = \bar{\mathrm{I}}[\sigma]$ for a certain σ strictly polynomial in $\mathscr{B}_0 \cup \langle\!\langle\mathscr{B}\rangle\!\rangle$.

Hint. Use 9.2.

Exercise 10.4. Prove that whenever a unary mapping Γ is t-recursive in $\mathscr{B}$, then

$$\Gamma = \lambda\theta.\bar{\mathrm{I}}[\varphi(I, \langle\!\langle\theta\rangle\!\rangle\bar{4}, \ldots, \langle\!\langle\theta\rangle\!\rangle\overline{n+3})]$$

with φ strictly polynomial in $\mathscr{B}_0 \cup \langle\!\langle\mathscr{B}\rangle\!\rangle$ and $n = c(\Sigma_4)c(\Sigma_7)$. In particular, $\Gamma = \lambda\theta.\bar{\mathrm{I}}[\varphi(I, \langle\!\langle\theta\rangle\!\rangle\bar{4})]$ provided $c(\Sigma_4) = c(\Sigma_7) = 1$.

Hint. Use 10.4, 9.8.

Under certain assumptions, a First t-Recursion Theorem can be proved using only $\mu\mathrm{A}_0$ and $\mathrm{t}\mu\mathrm{A}$.

Exercise 10.5. Let Σ_6 be prime recursive in $\mathscr{B}_0$ and suppose that $c(\Sigma_4) = c(\Sigma_6) = c(\Sigma_7) = 1$. Using $\mathrm{t}\mu\mathrm{A}$, prove that whenever a unary mapping Γ is t-recursive in $\mathscr{B}$, then the element $\mu\theta.\Gamma(\theta)$ exists and is t-recursive in $\mathscr{B}$.

Hint. The mapping $\lambda\theta.\langle\!\langle\Gamma(\theta)\rangle\!\rangle$ has a normal form $\lambda\theta.\bar{\mathrm{I}}[\varphi(I, \langle\!\langle\theta\rangle\!\rangle\bar{4})]$ with φ t-recursive in $\mathscr{B}$ by exercise 10.4. The mapping $\Gamma_1 = \lambda\theta.\Sigma_6(\bar{\mathrm{I}}[\varphi(I, \theta\bar{4})])$ is prime recursive in $\{\varphi\} \cup \mathscr{B}_0$ and $c(\Gamma_1) = 1$; hence the element $\theta_1 = \mu\theta.\Gamma_1(\theta)$ exists and is recursive in $\{\varphi\} \cup \mathscr{B}_0$ by 9.11. Following the proof of 10 8*, show that $\Sigma_5(\theta_1) = \mu\theta.\Gamma(\theta)$.

What happens if there is a second t-operation $\langle\!\langle\;\rangle\!\rangle^*$ over $\mathscr{F}$? Certainly, one would like to consider t*, t-recursiveness, i.e. to add both $\langle\!\langle\;\rangle\!\rangle, \langle\!\langle\;\rangle\!\rangle^*$ to the initial IOS-operations. One way of doing this is to proceed as in the present chapter with t-recursiveness playing the role of recursiveness and $\langle\!\langle\;\rangle\!\rangle^*$ playing the role of $\langle\!\langle\;\rangle\!\rangle$. There is also another option, namely to combine $\langle\!\langle\;\rangle\!\rangle, \langle\!\langle\;\rangle\!\rangle^*$ into a single t-operation, which allows a direct application of the theory already developed.

Exercise 10.6. Let $\langle\!\langle\;\rangle\!\rangle^*$ satisfy (0)–(5) with corresponding mappings $\Sigma_0^* - \Sigma_5^*$ t-recursive in a set $\mathscr{B}_0^*$ and suppose that there exists a mapping Σ^* t-recursive in $\mathscr{B}_0^*$ such that $\langle\!\langle\langle\!\langle\varphi\rangle\!\rangle\rangle\!\rangle^* = \Sigma^*(\langle\!\langle\varphi\rangle\!\rangle^*)$ for all φ. Show that $\langle\!\langle\;\rangle\!\rangle^{**} = \lambda\varphi.\langle\!\langle\langle\!\langle\varphi\rangle\!\rangle^*\rangle\!\rangle$ is a t-operation and there are mappings Γ, Γ^* recursive in $\mathscr{B}_0 \cup \langle\!\langle\mathscr{B}_0^*\rangle\!\rangle$ such that $\langle\!\langle\varphi\rangle\!\rangle = \Gamma(\langle\!\langle\varphi\rangle\!\rangle^{**})$, $\langle\!\langle\varphi\rangle\!\rangle^* = \Gamma^*(\langle\!\langle\varphi\rangle\!\rangle^{**})$ for all φ. (Therefore, t*, t-recursiveness and t**-recursiveness are equivalent.)

One drawback of the above construction is that if both $\langle\!\langle\;\rangle\!\rangle, \langle\!\langle\;\rangle\!\rangle^*$ satisfy corresponding $\mathrm{t}\mu$-axioms, then no such axiom for $\langle\!\langle\;\rangle\!\rangle^{**}$ seems to be automatically implied.

Exercise 10.7. Show that the implication in ($) can be replaced by its particular instance $\forall x(x\varphi \leq x\psi) \Rightarrow St(I)\varphi \leq St(I)\psi$, taking the equality $St(I)St(\varphi) = St(\varphi)$ as an axiom.

Exercise 10.8.** Let $\psi_0, \psi_1 \in \mathscr{F}$ satisfy $\bar{n}\psi_0 \in \mathscr{L}$, $\bar{n}\psi_0\psi_1 = \bar{n}$ for all n. Show that the operation $\langle\ \ \rangle = \lambda\varphi.(\varphi, St(\varphi))$ satisfies the assumptions of exercises 10.2–10.5 and so does the operation St, provided $I \in \mathscr{L} \circ \mathscr{F}$.

The following exercise establishes a 'boldface' t-Transition Theorem. As opposed to the '*lightface*' results, their '*boldface*' versions are concerned with a situation in which some or all members of $\mathscr{L}$ are taken as initial elements and used as enumeration indices. In other words, $\mathscr{L}$ plays the role of $\{\bar{n}/n \in \omega\}$.

Exercise 10.9. Let $K_4, K_5, K_6 \in \mathscr{F}$, $xK_4 = I$, $xK_5 = xx$, $xyK_6 = yx$ for all x, $y \in \mathscr{L}$ and allow in tμA segments of the form $\{\theta/x\theta \leq \tau\}$. Take $\langle\ \ \rangle = St$, an arbitrary member of $\mathscr{L}$ for K_3 and add $K_3 - K_6$ to the initial elements. Show that for every binary mapping Γ t-recursive in $\mathscr{B}$ there is a mapping Γ^* t-recursive in $\mathscr{B}$ such that $x\Gamma^*(\theta) = \Gamma(x, \theta)$ for all x, θ.

Hint. By induction on the construction of Γ.

Remark. These additional assumptions are satisfied by the operation St of exercise 10.1 and others to be considered later. One may use exercise 10.9 to get 'boldface' Normal Form, Enumeration, Second Recursion and Rice Theorems. For instance, given a finite $\mathscr{B}$ and not necessarily finite $\mathscr{L}_0 \subseteq \mathscr{L}$, there exists a σ t-recursive in $\mathscr{B}$ such that, if φ is t-recursive in $\mathscr{L}_0 \cup \mathscr{B}$, then $\varphi = x\bar{n}\sigma$ for certain $x \in cl(\mathscr{L}_0/\lambda xy.xyK_1)$, $n \in \omega$; similarly for mappings. (A related Enumeration Theorem is suggested by Skordev [1982a].) 'Boldface' representability results can also be obtained, with prime computability over $\mathscr{L}$ playing the role of μ-recursiveness. The First Recursion Theorem is not affected by the 'lightface–boldface' division of the theory. As for a 'boldface' Rogers Theorem, that is quite a different matter. And we can not refer to a 'boldface' Theory of Numberings, because there is no such theory available.

Exercise 10.10. Let $\mathscr{L}$ be a nonempty set of mappings $X: \mathscr{F} \to \mathscr{F}$, $St: \mathscr{F} \to \mathscr{F}$, K_0, K_1, $K_2 \in \mathscr{F}$ and suppose that the following axioms are satisfied.

sA$_1$. $X(K_0(\varphi, \psi)) = (X(\varphi), X(\psi))$.

sA$_2$. For all X, Y there is a Z such that

$$X(Y(K_1\varphi)) = Z(\varphi) \quad \text{and} \quad Z(K_2) = X(Y(I)).$$

($) $X(St(\varphi)\psi) = \varphi X(\psi), RX(I) = X(I)St(R),$

$$\forall X(\varphi X(\psi) \leq \rho X(\sigma)) \Rightarrow St(\varphi)\psi \leq St(\rho)\sigma.$$

Show that $\langle\ \ \rangle = \lambda\varphi.(\varphi, St(\varphi))$ is a t-operation with functional elements K_0, K_1, K_2.

Hint. Follow the proofs of 10.10–10.16.

The storing operation introduced in this way generalizes the former one which corresponds to a set $\mathscr{L}$ of mappings X such that $X(\varphi) = X(I)\varphi$ for all φ. Another example will be given in the exercises to chapter 21.

CHAPTER 11

The collection operation

This chapter is devoted to an infinitary pairing operation called *collection*. For instance, in example 3.1 the operation Co in question brings the members of a sequence $\{\varphi_n\}$ together in a single element $\varphi = Co(\{\varphi_n\})$ such that (in the notations of chapter 3) $\varphi(s01^n) = \varphi_n(s)$ for all n, s. Although this operation is elementary, being infinitary, it is classically noneffective.

One can reasonably argue that the present study of effective computability should have nothing to do with noneffective operations. However, collection has some interesting aspects. It is natural and also has effective applications. In particular, one may introduce the concept of a sequence $\{\varphi_n\}$ recursive in $\mathscr{B}$ as a sequence for which the element $Co(\{\varphi_n\})$ is recursive in $\mathscr{B}$. More important is the fact that essential parts of the recursion theory on IOS admit an adaptation with Co added to the initial operations; we are interested in such extensions anyway. Finally, while in chapter 2 the operations $\langle\quad\rangle$, $[\quad]$ were intuitively interpreted by expressions $(\varphi\bar{0}, \varphi\bar{1}, \varphi\bar{2}, \ldots)$, $(I, \varphi(I, \varphi(I, \ldots)))$, collection enables one to formally treat infinite expressions constructed by means of $\circ$, Π. In this respect, there is a clear parallel with the so called infinite diagrams in Scott [1971].

Given an IOS $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$, we say that $Co: \mathscr{F}^\omega \to \mathscr{F}$ is a *collection operation* iff the following three axioms are satisfied.

cA_1. Co is monotonic.

cA_2. $Co\{\varphi_n\}\psi = Co\{\varphi_n\psi\}$.

cA_3. $Co\{\varphi_n\} = (\varphi_0, Co\{\varphi_{n+1}\})$,

where $\{\varphi_{n+1}\}$ is the sequence $\varphi_1, \varphi_2, \ldots$ and $Co\{\varphi_n\}$ is written for $Co(\{\varphi_n\})$.

The axioms of Co are obviously infinitary versions of those of Π. A simple sufficient condition which ensures the existence of a collection operation will be given in the exercises, together with examples of IOS which satisfy it.

Some properties of collection follow, concerning in particular its interconnections with the initial IOS-operations.

Proposition 11.1. $\bar{n}Co\{\varphi_n\} = \varphi_n$ for all n.
This follows from axiom cA_3.

11.1 shows that the element $\sigma = Co\{\varphi_n\}$ is universal for $\{\varphi_n / n \in \omega\}$. Therefore, all countable subsets $\mathscr{B}$ of $\mathscr{F}$ have universal elements, though not necessarily in $\mathscr{B}$. Indeed, whenever σ is universal for $\{\bar{n}/n \in \omega\}$, then 4.4 easily implies $\sigma \neq \bar{n}$ for all n.

Proposition 11.2. $Co\{\varphi_n\bar{n}\}Co\{\psi_n\} = Co\{\varphi_n\psi_n\}$. In particular, $Co\{\bar{n}\}Co\{\varphi_n\} = Co\{\varphi_n\}$.

Proof. Using cA_2 and 11.1, we get

$$Co\{\varphi_n\bar{n}\}Co\{\psi_n\} = Co\{\varphi_n\bar{n}Co\{\psi_n\}\} = Co\{\varphi_n\psi_n\}.$$

Proposition 11.3. $Co\{(\varphi_n, \psi_n)\} = Co\{(\bar{n}L, \bar{n}R)\}(Co\{\varphi_n\}, Co\{\psi_n\})$.

Proof. It follows that

$$(\varphi_n, \psi_n) = (\bar{n}Co\{\varphi_n\}, \bar{n}Co\{\psi_n\}) = (\bar{n}L, \bar{n}R)(Co\{\varphi_n\}, Co\{\psi_n\})$$

for all n, which completes the proof by cA_2.

Proposition 11.4. $Co\{[\varphi_n]\} = Co\{(L, \overline{n+1})\}\ [\sigma]$, where $\sigma = Co\{\varphi_n(L, \overline{n+1})\}$.

Proof. Since $Co\{(L, \overline{n+1})\}[\sigma] = Co\{(I, \overline{n+1}[\sigma])\}$ and $Co\{[\varphi_n]\} = Co\{(I, R[\varphi_n])\}$, it suffices to show that $R[\varphi_n] = \overline{n+1}\ [\sigma]$ for all n.

The equality

$$\sigma(I, Co\{R[\varphi_n]\}) = Co\{\varphi_n(I, R[\varphi_n])\} = Co\{R[\varphi_n]\}$$

implies $R[\sigma] \leq Co\{R[\varphi_n]\}$ by 6.11, hence $\overline{n+1}\ [\sigma] \leq R[\varphi_n]$ for all n. On the other hand,

$$\varphi_n(I, \overline{n+1}\ [\sigma]) = \varphi_n(L, \overline{n+1})\ [\sigma] = \overline{n+1}\ [\sigma],$$

hence $R[\varphi_n] \leq \overline{n+1}\ [\sigma]$ for all n by 6.11. This completes the proof.

The above properties indicate that collection behaves to some extent as a t-operation. The following statement shows that consecutive applications of Co can be reduced to a single one.

Proposition 11.5. Let $J{:}\omega^2 \to \omega$ be a bijection, $\rho = Co\{Co\{\overline{J(m,n)}\}_n\}_m$ and $\varphi_{J(m,n)} = \varphi_{m,n}$ for all m, n. Then $Co\{Co\{\varphi_{m,n}\}_n\}_m = \rho Co\{\varphi_k\}$.

Proof.

$$Co\{Co\{\varphi_{m,n}\}_n\}_m = Co\{Co\{\varphi_{J(m,n)}\}_n\}_m = Co\{Co\{\overline{J(m,n)}Co\{\varphi_k\}\}_n\}_m = \rho Co\{\varphi_k\}.$$

Proposition 11.6. $\Delta(\varphi, \psi) = \bar{1}[\sigma]$, where $\sigma = Co\{(\varphi\psi^n L, \overline{n+2})\}$.

Proof. We have

$$\begin{aligned}\sigma(I, Co\{\Delta(\varphi, \psi)\psi^n\}) &= Co\{(\varphi\psi^n, \Delta(\varphi, \psi)\psi^{n+1})\} = Co\{(\varphi, \Delta(\varphi, \psi)\psi)\psi^n\}\\ &= Co\{\Delta(\varphi, \psi)\psi^n\},\end{aligned}$$

hence $R[\sigma] \leq Co\{\Delta(\varphi, \psi)\psi^n\}$ by 6.11. In particular, $\bar{1}[\sigma] \leq \Delta(\varphi, \psi)$. On the other hand,

$$\sigma(\psi, R^2[\sigma]) = \sigma(\psi L, R^2)[\sigma] = R\sigma[\sigma] = R^2[\sigma]$$

implies $R[\sigma]\psi \leq R^2[\sigma]$ by 6.11 again. Therefore,

$$(\varphi, \bar{1}[\sigma]\psi) \leq (\varphi, \bar{2}[\sigma]) = (\varphi\bar{0}, \bar{2})[\sigma] = \bar{0}\sigma[\sigma] = \bar{1}[\sigma],$$

hence $\Delta(\varphi, \psi) \leq \bar{1}[\sigma]$. The proof is complete.

Primitive recursion can also be expressed by the equality $\Delta(\varphi,\psi)=\langle I\rangle Co\{\varphi\psi^n\}$ which follows easily from 6.34. Conversely, $Co\{\varphi\psi^n\}=Co\{\bar{n}\}\Delta(\varphi,\psi)$ follows by making use of cA_2.

The notion of relative recursiveness is naturally extended by adding Co to the initial IOS-operations. Namely, φ is c-*recursive in* $\mathscr{B}$ iff

$$\varphi\in cl(\{L,R\}\cup\mathscr{B}/\circ,\Pi,\langle\quad\rangle,[\quad],Co).$$

Proposition 6.9, 11.6 imply that, equivalently, φ is c-recursive in $\mathscr{B}$ iff $\varphi\in cl(\{L,A\}\cup\mathscr{B}/\circ,[\quad],Co)$. Notions of elements *prime* c-*recursive*, *primitive* c-*recursive*, c-*primitive* and c-*polynomial in* $\mathscr{B}$ are introduced in a similar way. It follows from 11.6 that the relative c-recursiveness and prime c-recursiveness are equivalent, while the equality $\Delta(\varphi,\psi)=\langle I\rangle Co\{\varphi\psi^n\}$ implies that φ is primitive c-recursive in $\mathscr{B}$ iff it is c-polynomial in $\{\langle I\rangle\}\cup\mathscr{B}$. The ordinary properties from the beginning of chapter 7 arise again with 7.4 modified essentially as follows.

Proposition 11.7. If φ is c-recursive (c-polynomial) in $\mathscr{B}$, then φ is c-recursive (respectively, c-polynomial) in a countable subset of $\mathscr{B}$.

It is worth mentioning that, while there are $\max\{\mathrm{Card}(\mathscr{B}),\omega\}$ elements recursive in $\mathscr{B}$, there are $\mathrm{Card}(\{L,R\}\cup\mathscr{B})^\omega$ elements c-recursive in $\mathscr{B}$ since $Co\{\varphi_n\}=Co\{\psi_n\}$ iff $\forall n(\varphi_n=\psi_n)$.

The classically noneffective nature of Co finds its expression in the fact that all partial number theoretic functions are representable in terms of Co.

Proposition 11.8. Every partial number theoretic function f is represented by an element $\varphi\in cl(\mathscr{D}\cup\{O\}/Co)$.

Proof. By induction on the arity of f.

If f is unary, then $Co\{\overline{f(n)}\}$ represents it. (Recall our convention that $\overline{f(n)}=O$ whenever $f(n)\uparrow$.)

Let f be $n+1$-ary and for all m an element φ_m represent $\lambda s_1\ldots s_n.f(s_1,\ldots,s_n,m)$ by the induction clause. Then $\varphi=Co\{\varphi_m\}$ represents f since

$$\bar{s}_1\ldots\bar{s}_n\bar{m}\varphi=\bar{s}_1\ldots\bar{s}_n\varphi_m=\overline{f(s_1,\ldots,s_n,m)}$$

for all $s_1,\ldots,s_n$, m.

Passing from elements to mappings, one sees that collection makes it possible to consider ω-ary mappings. A notion of mapping $\Gamma:\mathscr{F}^\omega\to\mathscr{F}$ c-*recursive in* $\mathscr{B}$ is introduced inductively as follows, taking in view of 6.9, 11.6 the element A initial and omitting the operations Π, $\langle\quad\rangle$.

1. The mappings $\Gamma=\lambda\{\theta_n\}.\theta_i$, $i\geq 0$ and $\Gamma=\lambda\{\theta_n\}.\psi$, $\psi\in\{L,A\}\cup\mathscr{B}$, are c-recursive in $\mathscr{B}$

2. If $\Gamma_m:\mathscr{F}^\omega\to\mathscr{F}$ are c-recursive in $\mathscr{B}$ for all m, then so are

$$\Gamma=\lambda\{\theta_n\}.\Gamma_0\{\theta_n\}\Gamma_1\{\theta_n\},\quad \Gamma=\lambda\{\theta_n\}.[\Gamma_0\{\theta_n\}],$$
$$\Gamma=\lambda\{\theta_n\}.Co\{\Gamma_m\{\theta_n\}\}.$$

Of course, n-ary mappings c-recursive in $\mathscr{B}$ may be considered as well: an ω-ary mapping with the clause $\lambda\{\theta_n\}.\theta_i$ used for finitely many i in its construction has only finitely many genuine arguments.

The properties of the collection operation given above allow us to establish a Normal c-Form Theorem.

Proposition 11.9. If $\Gamma:\mathscr{F}^{\omega}\to\mathscr{F}$ is c-recursive in $\mathscr{B}$, then

$$\Gamma=\lambda\{\theta_n\}.\bar{1}[\varphi(I,Co\{\theta_{i_n}\bar{n}\})]$$

with a certain sequence $\{i_n\}$ and φ c-polynomial in $\mathscr{B}$. Another normal form is $\Gamma=\lambda\{\theta_n\}.\bar{1}[\varphi(I,Co\{\langle\theta_n\rangle\})]$.

Proof. We shall prove first that

$$\Gamma=\lambda\{\theta_n\}.\psi[\varphi(I,Co\{\theta_{i_n}\bar{n}\}\chi)]$$

with certain ψ, φ, χ c-polynomial in $\mathscr{B}$.

If $\Gamma=\lambda\{\theta_n\}.\theta_i$, then

$$\Gamma=\lambda\{\theta_n\}.R[\overline{i+1}(I,Co\{\theta_n\bar{n}\}Co\{L\})],$$

$\{L\}$ standing for the sequence L, $L,\ldots$.

If $\Gamma=\lambda\{\theta_n\}.\psi$, $\psi\in\{L,A\}\cup\mathscr{B}$, then $\Gamma=\lambda\{\theta_n\}.R[\psi L^2(I,Co\{\theta_n\bar{n}\})]$.

Let $\Gamma_m=\lambda\{\theta_n\}.\psi_m[\varphi_m(I,Co\{\theta_{i_{m,n}}\bar{n}\}_n\chi_m)]$ and ψ_m,φ_m,χ_m be c-polynomial in $\mathscr{B}$ for all m.

If $\Gamma=\lambda\{\theta_n\}.\Gamma_0\{\theta_n\}\Gamma_1\{\theta_n\}$, then we get by 6.15 elements ψ, φ', χ', χ'' c-polynomial in $\mathscr{B}$ such that

$$\Gamma=\lambda\{\theta_n\}.\psi[\varphi'(I,Co\{\theta_{i_{0,n}}\bar{n}\}\chi',Co\{\theta_{i_{1,n}}\bar{n}\}\chi'')].$$

Taking $\varphi=\varphi'(L,Co\{\overline{2n+1}\},\ Co\{\overline{2n+2}\})$, $\chi=Co(\bar{0}\chi',\bar{0}\chi'',\bar{1}\chi',\bar{1}\chi'',\ldots)$ and $j_n=i_{\mathrm{rem}(n,2),n}$ for all n, we get

$$\Gamma=\lambda\{\theta_n\}.\psi[\varphi(I,Co\{\theta_{j_n}\bar{n}\}\chi)].$$

The case of iteration is a bit simpler, using 6.16 instead of 6.15.

If $\Gamma=\lambda\{\theta_n\}.Co\{\Gamma_m\{\theta_n\}\}$, then we get by 11.4 elements ψ, φ', $\chi_{m,n}$ c-polynomial in $\mathscr{B}$ for all m, n such that $\Gamma=\lambda\{\theta_n\}.\psi[\varphi'(I,Co\{Co\{\theta_{i_{m,n}}\chi_{m,n}\}_n\}_m)]$. Take J, ρ as in 11.5, $j_{J(m,n)}=i_{m,n}$, $\sigma_{J(m,n)}=\chi_{m,n}$ for all m, n and $\chi=Co\{\sigma_k\}$. It follows by 11.5 that

$$Co\{Co\{\theta_{i_{m,n}}\chi_{m,n}\}_n\}_m=\rho Co\{\theta_{j_k}\sigma_k\}_k=\rho Co\{\theta_{j_k}\bar{k}\}\chi$$

for all $\{\theta_n\}$. Therefore, $\Gamma=\lambda\{\theta_n\}.\psi[\varphi(I,Co\{\theta_{j_k}\bar{k}\}\chi)]$ with $\varphi=\varphi'(L,\rho R)$.

Now let $\Gamma=\lambda\{\theta_n\}.\psi[\varphi(I,Co\{\theta_{i_n}\bar{n}\}\chi)]$ with ψ, φ, χ c-polynomial in $\mathscr{B}$. Substituting $\varphi(L,Co\{\theta_{i_n}\overline{n+1}\})[\chi L]$ for $\varphi(I,Co\{\theta_{i_n}\bar{n}\}\chi)$ and using 6.16, 6.14, we almost get the desired normal form except that $Co\{\theta_{i_n}\overline{n+5}\}$ appears instead of $Co\{\theta_{i_n}\bar{n}\}$. However, $Co\{\theta_{i_n}\overline{n+5}\}=R^5Co\{\theta_{j_n}\bar{n}\}$, provided $j_{n+5}=i_n$ for all n.

The alternative normal form is obtained immediately by

$$Co\{\theta_{i_n}\bar{n}\}=Co\{\bar{n}\bar{i}_n\}Co\{\langle\theta_n\rangle\}.$$

This completes the proof.

A finitary version of 11.9 gives normal forms

$$\Gamma=\lambda\theta_1\ldots\theta_n.\bar{1}[\varphi(I,Co\{\theta_{i_m}\bar{m}\})],$$

$1 \le i_m \le n$ for all m, or

$$\Gamma = \lambda\theta_1 \ldots \theta_n . \bar{\mathrm{I}}[\varphi(I, \langle\theta_1\rangle, \ldots, \langle\theta_n\rangle)]$$

with φ c-polynomial in $\mathscr{B}$. The elements c-recursive in $\mathscr{B}$ have a normal form $\bar{\mathrm{I}}[\sigma]$ with σ c-polynomial in $\mathscr{B}$.

A First c-Recursion Theorem follows.

Proposition 11.10*. If $\Gamma:\mathscr{F}^{\omega} \to \mathscr{F}$ is c-recursive in $\mathscr{B}$, then there is a mapping $\Gamma_0:\mathscr{F}^{\omega} \to \mathscr{F}$ c-recursive in $\mathscr{B}$ such that $\Gamma_0 = \lambda\{\theta_n\}.\mu\theta.\Gamma(\theta, \{\theta_n\})$. (That is, for all $\{\theta_n\}$ $\Gamma(\Gamma_0\{\theta_n\}, \{\theta_n\}) = \Gamma_0\{\theta_n\}$ and whenever $\Gamma(\tau, \{\theta_n\}) \le \tau$, then $\Gamma_0\{\theta_n\} \le \tau$.)

Proof. The Normal c-Form Theorem gives

$$\Gamma = \lambda\{\theta_n\}.\bar{\mathrm{I}}[\varphi(I, \langle\theta_0\rangle, Co\{\langle\theta_{n+1}\rangle\})]$$

for a certain φ c-recursive in $\mathscr{B}$. The mapping $\Gamma^* = \lambda\theta\theta_1.\bar{\mathrm{I}}[\varphi(I, \langle\theta\rangle, \theta_1)]$ is recursive in φ, hence so is $\Gamma^{**} = \lambda\theta_1.\mu\theta.\Gamma^*(\theta, \theta_1)$ by 9.14*. Therefore, the mapping $\Gamma_0 = \lambda\{\theta_n\}.\ \Gamma^{**}(Co\{\langle\theta_n\rangle\})$ is c-recursive in $\mathscr{B}$. It follows that

$$\begin{aligned}\Gamma(\Gamma_0\{\theta_n\}, \{\theta_n\}) &= \Gamma^*(\Gamma_0\{\theta_n\}, Co\{\langle\theta_n\rangle\}) \\ &= \Gamma^*(\Gamma^{**}(Co\{\langle\theta_n\rangle\}), Co\{\langle\theta_n\rangle\}) \\ &= \Gamma^{**}(Co\{\langle\theta_n\rangle\}) = \Gamma_0\{\theta_n\}.\end{aligned}$$

Whenever $\Gamma(\tau, \{\theta_n\}) \le \tau$, then

$$\Gamma^*(\tau, Co\{\langle\theta_n\rangle\}) = \Gamma(\tau, \{\theta_n\}) \le \tau,$$

hence $\Gamma^{**}(Co\{\langle\theta_n\rangle\}) \le \tau$, i.e., $\Gamma_0\{\theta_n\} \le \tau$. The proof is complete.

A finitary version of 11.10* states that whenever $\Gamma:\mathscr{F}^{n+1} \to \mathscr{F}$ is c-recursive in $\mathscr{B}$, then the mapping $\lambda\theta_1 \ldots \theta_n.\mu\theta.\Gamma(\theta_1, \ldots, \theta_n, \theta)$ exists and is c-recursive in $\mathscr{B}$. In particular, the element $\mu\theta.\Gamma(\theta)$ exists and is c-recursive in $\mathscr{B}$, provided $n = 0$.

The collection operation also enables us to solve infinite systems of inequalities.

Proposition 11.11*. Let $\Gamma_m:\mathscr{F}^{\omega} \to \mathscr{F}$ be c-recursive in $\mathscr{B}$ for all m. Then there is a sequence $\{\varphi_n\}$ of elements c-recursive in $\mathscr{B}$ which is the least solution of the system $\Gamma_m\{\theta_n\} \le \theta_m$, $m \ge 0$.

Proof. The mapping $\Gamma = \lambda\theta.Co\{\Gamma_m\{\bar{n}\theta\}\}$ is c-recursive in $\mathscr{B}$; hence so is the element $\varphi = \mu\theta.\Gamma(\theta)$ by 11.10*. Taking $\varphi_n = \bar{n}\varphi$, we get

$$\Gamma_m\{\varphi_n\} = \Gamma_m\{\bar{n}\varphi\} = \bar{m}\Gamma(\varphi) = \bar{m}\varphi = \varphi_m$$

for all m.

If $\Gamma_m\{\tau_n\} \le \tau_m$ for all m, then

$$\Gamma(Co\{\tau_n\}) = Co\{\Gamma_m\{\tau_n\}\} \le Co\{\tau_m\},$$

hence $\varphi \le Co\{\tau_n\}$, which implies $\varphi_n \le \tau_n$ for all n. (Notice that this is the first and only time cA_1 was used. Without this axiom one would get that $\{\varphi_n\}$ is a least solution to the corresponding system of equalities.) The proof is complete.

The mappings Γ_m may have finitely many arguments. For instance, a simple infinite system of inequalities is $\Gamma_m(\theta_{m+1}) \leq \theta_m$, $m \geq 0$. A parametrized version of 11.11* concerns systems of the form $\Gamma_{2m}\{\theta_n\} \leq \theta_{2m}$, $m \geq 0$.

This concludes our discussion of the collection operation. As compared with similar considerations in the previous two chapters, there is a noticeable omission. Despite having a First c-Recursion Theorem, no c-Enumeration Theorem has been established. In chapter 9 an element σ was universal for a countable set $\mathscr{U}$ iff $\mathscr{U} \subseteq \{\bar{n}\sigma/n\in\omega\}$; given a finite set $\mathscr{B}$, there were countably many elements recursive in $\mathscr{B}$. Now, given a countable set $\mathscr{B}$, there are 2^ω elements c-recursive in $\mathscr{B}$; hence the elements $\bar{n}$, $n\in\omega$ are inadequate as enumeration indices. We shall return to this problem in the exercises.

As mentioned in the introduction to this chapter, *infinite expressions* can be given a formal interpretation by using collection operation. We sketch the idea and give several examples.

Given a sequence $\{\varphi_n\}$, what value, if any, should the infinite expression $(I, \varphi_0(I, \varphi_1(I, \ldots)))$ have? One would like it to be a member of $\mathscr{F}$, of course. Suppose for a moment that it is σ_0 and similarly $\sigma_{n+1} = (I, \varphi_{n+1}(I, \varphi_{n+2}(I, \ldots)))$ for all n. Then it follows immediately that the sequence $\{\sigma_n\}$ satisfies the system

$$(1) \qquad (I, \varphi_n\theta_{n+1}) = \theta_n, \quad n \geq 0.$$

However, this system of equalities actually does have solutions. Proposition 11.11* ensures that (1) has a least solution which is a sequence consisting of elements c-recursive in $\{\varphi_n\}$. Now one can return to formally define σ_n for all n. Namely, take $\{\sigma_n\}$ to be the least solution of (1); then σ_0 is the value of the expression we are interested in. The correctness of this definition needs proof since equivalent systems of equalities can be attached to an expression in various ways.

Therefore, arbitrary infinite expressions can be interpreted by means of systems of equalities and vice versa, single equalities and systems of equalities yield infinite expressions by consecutively replacing right sides of equalities by their left ones. For instance, given the equality $(I, \varphi\theta) = \theta$, we get $(I, \varphi(I, \varphi(I, \ldots)))$ by consecutively substituting $(I, \varphi\theta)$ for θ. Other basic operations besides $\circ$, Π may also be used.

If $\varphi_n = \varphi$ for all n and $\{\sigma_n\}$ is the least solution of (1), then it is quite immediate that $\sigma_n = \sigma_0$ for all n. Therefore, σ_0 is the least solution to $(I, \varphi\theta) = \theta$, i.e. $\sigma_0 = [\varphi]$. So one may write $[\varphi] = (I, \varphi(I, \varphi(I, \ldots)))$. The equality $\langle\varphi\rangle = (\varphi\bar{0}, \varphi\bar{1}, \ldots)$ is verified in exercise 11.6 below.

Let $\{\sigma_n\}$ be the least solution of the system $(\varphi_n, \theta_{n+1}) = \theta_n$, $n \geq 0$. Then $\sigma_0 = (\varphi_0, \varphi_1, \ldots)$ and it easily follows that $Co\{\varphi_n\} = Co\{\bar{n}\}\sigma_0$, hence collection itself is expressed via an infinite expression.

EXERCISES TO CHAPTER 11

Exercise 11.1. Let $\mathscr{S}$ be an OS to satisfy the following condition.

($c*$) The semigroup $\mathscr{F}$ has a least element O and for any increasing sequence $\{\theta_n\}$ there is a θ such that $\theta\psi = \sup_n(\theta_n\psi)$ and $(\psi, \theta) = \sup_n(\psi, \theta_n)$ for all ψ.

Determine an operation Co satisfying the axioms $cA_1 - cA_3$.

Hint. Given a sequence $\{\varphi_n\}$, define $Co\{\varphi_n\}$ as $\sup_n \theta_n$, where $\theta_0 = O$ and $\theta_{n+1} = (\varphi_0, \varphi_1, \ldots, \varphi_n, O)$.

Notice that $Co\{\varphi_n\} = (\varphi_0, \varphi_1, \ldots)$ and $\langle I \rangle = Co\{\bar{n}\}$ in such a case. It follows by what we know about the IOS of examples 4.7, 4.8, 3.2 that those spaces satisfy $(c*)$.

Exercise 11.2. Show that $\Gamma : \mathscr{F}^\omega \to \mathscr{F}$ is c-recursive in $\{\psi_n\}$ iff there is a c-recursive mapping $\Gamma^* : \mathscr{F}^\omega \to \mathscr{F}$ such that

$$\Gamma = \lambda\{\theta_n\}.\Gamma^*(\psi_0, \theta_0, \psi_1, \theta_1, \ldots).$$

Exercise 11.3. Show that $(\varphi_0, \varphi_1, \ldots) = \bar{1}[\rho]Co\{\varphi_n\}$, where $\rho = Co\{(\bar{n}L, \overline{n+2})\}$.

Hint. By definition $(\varphi_0, \varphi_1, \ldots) = \sigma_0$, where $\{\sigma_n\}$ is the least solution to the system $(\varphi_n, \theta_{n+1}) = \theta_n$, $n \geq 0$. Following the proof of 11.11*, show that $\sigma_n = \bar{n}\sigma$ for all n, where $\sigma = \mu\theta.Co\{(\varphi_n, \overline{n+1}\,\theta)\} = \mu\theta.\rho(Co\{\varphi_n\}, \theta) = R[\rho]Co\{\varphi_n\}$.

Exercise 11.4. Show that $(\varphi_0, \varphi_1, \ldots)\psi = (\varphi_0\psi, \varphi_1\psi, \ldots)$.

Hint. Make use of the previous exercise.

Exercise 11.5. Show that $(I, \varphi_0(I, \varphi_1(I, \ldots))) = \bar{1}[\rho]$, where $\rho = Co\{(L, \overline{\varphi_n n+2})\}$.

Hint. See the hint to exercise 11.3.

Exercise 11.6. Show that $\Delta(\varphi, \psi) = (\varphi, \varphi\psi, \varphi\psi^2, \ldots)$. In particular, $\langle \varphi \rangle = (\varphi\bar{0}, \varphi\bar{1}, \ldots)$.

Hint. The sequence $\{\Delta(\varphi, \psi)\psi^n\}$ is a solution of the system $(\varphi\psi^n, \theta_{n+1}) = \theta_n$, $n \geq 0$; hence $\sigma_0 = (\varphi, \varphi\psi, \ldots) \leq \Delta(\varphi, \psi)$. Use exercise 11.4 to show that $(\varphi, \sigma_0\psi) = \sigma_0$, which implies $\Delta(\varphi, \psi) \leq \sigma_0$.

Further to the enumeration problem for c-recursiveness, let $\mathscr{S}$ be augmented by a storing operation St satisfying the assumptions of exercise 10.9. Let $K_7 \in \mathscr{F}$, $\varnothing \subset \mathscr{L}_0 \subseteq \mathscr{L}$ and suppose that for every sequence $\{x_n\bar{k}_n\}_n$, $x_n \in \mathscr{L}_0$, there exist $x \in \mathscr{L}_0$, $k \in \omega$ such that $x\bar{k}K_7 = Co\{x_n\bar{k}_n\}$. (This imposes additional constraints, e.g. $\text{Card}(M) \geq 2^\omega$ in examples 4.7, 4.8.) Assume also that $\mathscr{L}_0^2 K_1 \subseteq \mathscr{L}_0$. Adding $K_0 - K_7$ to the initial elements and Co, St to the initial operations, the notion of recursiveness is then extended to t, c-recursiveness for which the following 'boldface' t, c-Enumeration Theorem can be established.

Exercise 11.7. Show under the above assumptions that for any countable $\mathscr{B} \subseteq \mathscr{F}$ there is an element σ t, c-recursive in $\mathscr{B}$ such that whenever φ is t, c-recursive in $\mathscr{L}_0 \cup \mathscr{B}$, then $\varphi = x\bar{n}\sigma$ for certain $x \in \mathscr{L}_0$, $n \in \omega$.

Hint. Modify the proof of 9.18, using an analogue to 10.2. Alternatively, follow the idea of exercise 9.5.

As in the case of t-recursiveness discussed after exercise 10.9, 'boldface' Second Recursion and Rice Theorems can also be established.

CHAPTER 12

Consecutive spaces

Given an OS $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$, we have so far studied elements 'computable' from certain elements by means of the initial operations $\circ, \Pi$ plus $\langle\ \ \rangle, [\ \ ]$, provided $\mathscr{S}$ is iterative, and possibly $\langle\!\langle\ \ \rangle\!\rangle$, Co. To accommodate notions such as Kleene's [1955] concept of hyperarithmetical function, however, one should be able to consider elements 'computable' from certain operations, particularly operations involving quantifiers.

This leads to the idea of trying to construct another OS $\mathscr{S}' = (\mathscr{F}', I', \Pi', L', R')$ the carrier $\mathscr{F}'$ of which would consist of mappings over $\mathscr{F}$. It is an important feature of the algebraic system of OS that such a construction can be carried out successfully, and chapters 12, 13 are concerned with recursion theory on pairs of spaces $\mathscr{S}, \mathscr{S}'$.

Proposition 12.1. Take $\mathscr{F}' = \{\varphi'/\varphi' : \mathscr{F} \to \mathscr{F}\ \&\ \varphi'$ is monotonic$\}$, $\varphi' \leq \psi'$ iff $\forall\theta(\varphi'(\theta) \leq \psi'(\theta))$, $\varphi'\psi' = \lambda\theta.\varphi'(\psi'(\theta))$, $(\varphi', \psi') = \lambda\theta.(\varphi'(\theta),\ \psi'(\theta))$, $I' = \lambda\theta.\theta$, $L' = \lambda\theta.L\theta$ and $R' = \lambda\theta.R\theta$. Then $\mathscr{S}' = (\mathscr{F}', I', \Pi', L', R')$ is an OS.

Proof. We have

$$I'\varphi'(\theta) = I'(\varphi'(\theta)) = \varphi'(\theta),$$
$$\varphi' I'(\theta) = \varphi'(I'(\theta)) = \varphi'(\theta),$$
$$\varphi'(\psi'\chi')(\theta) = \varphi'(\psi'(\chi'(\theta))) = (\varphi'\psi')\chi'(\theta)$$

for all θ, hence $I'\varphi' = \varphi' I' = \varphi'$ and $\varphi'(\psi'\chi') = (\varphi'\psi')\chi'$.

If $\varphi' \leq \varphi'_1, \psi' \leq \psi'_1$, then

$$\varphi'\psi'(\theta) = \varphi'(\psi'(\theta)) \leq \varphi'(\psi'_1(\theta)) \leq \varphi'_1(\psi'_1(\theta)) = \varphi'_1\psi'_1(\theta)$$

for all θ, hence $\varphi'\psi' \leq \varphi'_1\psi'_1$. Therefore, $\mathscr{F}'$ is a partially ordered semigroup with unit I'.

If $\varphi' \leq \varphi'_1,\ \psi' \leq \psi'_1$, then

$$(\varphi', \psi')(\theta) = (\varphi'(\theta), \psi'(\theta)) \leq (\varphi'_1(\theta), \psi'_1(\theta)) = (\varphi'_1, \psi'_1)(\theta)$$

for all θ; hence $(\varphi', \psi') \leq (\varphi'_1, \psi'_1)$. It also follows that

$$(\varphi', \psi')\chi'(\theta) = (\varphi', \psi')(\chi'(\theta)) = (\varphi'(\chi'(\theta)),\ \psi'(\chi'(\theta)))$$
$$= (\varphi'\chi'(\theta), \psi'\chi'(\theta)) = (\varphi'\chi', \psi'\chi')(\theta)$$

for all θ; hence $(\varphi', \psi')\chi' = (\varphi'\chi', \psi'\chi')$. Finally,

$$L'(\varphi', \psi')(\theta) = L'((\varphi'(\theta), \psi'(\theta))) = L(\varphi'(\theta), \psi'(\theta)) = \varphi'(\theta)$$

implies $L'(\varphi', \psi') = \varphi'$. Similarly, $R'(\varphi', \psi') = \psi'$, which completes the proof.

Sometimes one may be interested in a smaller space $\mathscr{S}'$ composed of mappings which are not only monotonic but say, continuous. That is why we introduce the notion of consecutive spaces as follows.

Let $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ and $\mathscr{S}' = (\mathscr{F}', I', \Pi', L', R')$ be OS such that $\varphi' : \mathscr{F} \to \mathscr{F}$ for all $\varphi' \in \mathscr{F}'$, $\varphi' \leq \psi'$ iff $\forall \theta(\varphi'(\theta) \leq \psi'(\theta))$ and $(\varphi', \psi') = \lambda\theta.(\varphi'(\theta), \psi'(\theta))$. We write $\bar{\varphi} = \lambda\theta.\varphi\theta$, $\check{\varphi} = \lambda\theta.\varphi$ and $(\text{———})\check{}$ for $\lambda\theta.\text{———}$, provided θ does not occur in the expression ———, $Id = \check{I}$, $Ml = \lambda\theta.L\theta R\theta$, $\bar{\mathscr{B}} = \{\bar{\psi}/\psi \in \mathscr{B}\}$ and $\check{\mathscr{B}} = \{\check{\psi}/\psi \in \mathscr{B}\}$, where $\varphi \in \mathscr{F}$, $\mathscr{B} \subseteq \mathscr{F}$. Let also $\bar{\mathscr{F}} \subseteq \mathscr{F}'$ and $Id, Ml \in \mathscr{F}'$. Then $\mathscr{S}, \mathscr{S}'$ are said to be *consecutive* OS. (For such spaces appear consecutively in the hierarchies of OS studied in chapter 15.)

The OS $\mathscr{S}, \mathscr{S}'$ of 12.1 are consecutive since the mappings $\bar{\varphi}$, Id, Ml are monotonic, and hence in $\mathscr{F}'$. Similar constructions yielding consecutive spaces will be given in chapter 19, while particular consecutive spaces will be studied in chapters 28–30.

Several corollaries to the above definition follow, dashed letters standing for members of $\mathscr{F}'$.

Proposition 12.2. $I' = \bar{I}$. Therefore, $I' = \lambda\theta.\theta$.

Proof.

$$I' = \lambda\theta.I'(\theta) = \lambda\theta.I'(\bar{I}(\theta)) = I'\bar{I} = \bar{I}.$$

Proposition 12.3. $\bar{\varphi}Id = \check{\varphi}$.

Proof.

$$\bar{\varphi}Id = \lambda\theta.\bar{\varphi}(Id(\theta)) = \lambda\theta.\bar{\varphi}(I) = \lambda\theta.\varphi = \check{\varphi}.$$

Notice that 12.3 implies $\check{\mathscr{F}} \subseteq \mathscr{F}'$.

Proposition 12.4. $\check{\varphi}\psi' = \check{\varphi}$ for all φ, ψ'. In particular, $Id\varphi' = Id$ for all φ'.

Proof.

$$\check{\varphi}\psi' = \lambda\theta.\check{\varphi}(\psi'(\theta)) = \lambda\theta.\varphi = \check{\varphi}.$$

Proposition 12.5. $Id \notin \bar{\mathscr{F}}$.

Proof. Suppose that $Id = \bar{\varphi}$. Then $\varphi = \bar{\varphi}(I) = Id(I) = I$, hence $Id = \bar{I} = I'$. Therefore,

$$L' = I'L' = IdL' = Id = I',$$

which is not the case.

Proposition 12.6. $\check{\varphi} \leq \check{\psi}$ iff $\varphi \leq \psi$.

Proposition 12.7. The members of $\mathscr{F}'$ are monotonic mappings.

Proof. Let $\varphi' \in \mathscr{F}'$ and $\theta \leq \tau$. Then $\check{\theta} \leq \check{\tau}$; hence $\varphi'\check{\theta} \leq \varphi'\check{\tau}$, which implies $\varphi'(\theta) \leq \varphi'(\tau)$.

Proposition 12.8. $\bar{\varphi} \leq \bar{\psi}$ iff $\varphi \leq \psi$.

Proof. If $\varphi \leq \psi$, then $\bar{\varphi}(\theta) = \varphi\theta \leq \psi\theta = \bar{\psi}(\theta)$ for all θ, hence $\bar{\varphi} \leq \bar{\psi}$. If $\bar{\varphi} \leq \bar{\psi}$, then $\varphi = \bar{\varphi}(I) \leq \bar{\psi}(I) = \psi$.

Proposition 12.9. $\bar{\varphi}\bar{\psi} = \overline{\varphi\psi}$.

Proof.

$$\bar{\varphi}\bar{\psi} = \lambda\theta.\bar{\varphi}(\bar{\psi}(\theta)) = \lambda\theta.\varphi\psi\theta = \overline{\varphi\psi}.$$

Proposition 12.10. $(\bar{\varphi}, \bar{\psi}) = \overline{(\varphi, \psi)}$.

Proof.

$$(\bar{\varphi}, \bar{\psi}) = \lambda\theta.(\bar{\varphi}(\theta), \bar{\psi}(\theta)) = \lambda\theta.(\varphi\theta, \psi\theta) = \lambda\theta.(\varphi, \psi)\theta = \overline{(\varphi, \psi)}.$$

Since $\bar{L}(\varphi', \psi') = \varphi'$ and $\bar{R}(\varphi', \psi') = \psi'$ for all φ', ψ', one may also assume that $L' = \bar{L}$ and $R' = \bar{R}$. Therefore, the following statement holds.

Proposition 12.11. The OS $\mathscr{S}$ is isomorphic with the proper subspace

$\bar{\mathscr{S}} = (\bar{\mathscr{F}}, \bar{I}, \Pi' \restriction \bar{\mathscr{F}}^2, \bar{L}, \bar{R})$ of $\mathscr{S}'$.

Some immediate statements follow.

a **Proposition 12.12.** $\bar{\varphi}\psi' = \lambda\theta.\varphi\psi'(\theta)$.

b **Proposition 12.13.** $\varphi'\bar{\psi} = \lambda\theta.\varphi'(\psi\theta)$.

c **Proposition 12.14.** $\varphi' Id = (\varphi'(I))\check{}$.

d **Proposition 12.15.** $\varphi'\check{\psi} = (\varphi'(\psi))\check{}$. In particular, $\bar{\varphi}\check{\psi} = (\varphi\psi)\check{}$.

Proposition 12.16. $Ml(\varphi', \psi') = \lambda\theta.\varphi'(\theta)\psi'(\theta)$.

Proof.

$$Ml(\varphi', \psi') = \lambda\theta.Ml((\varphi', \psi')(\theta)) = \lambda\theta.Ml((\varphi'(\theta), \psi'(\theta))) = \lambda\theta.\varphi'(\theta)\psi'(\theta).$$

Therefore, the element Ml represents in $\mathscr{F}'$ the semigroup multiplication of $\mathscr{F}$.

Proposition 12.17. $Ml \notin \bar{\mathscr{F}}$.

Proof. Suppose that $Ml = \bar{\varphi}$. Then $\varphi = \bar{\varphi}(I) = Ml(I) = LR$, hence $LR^2 = \overline{LR}(R) = Ml(R) = LR^3$, which is not the case.

Proposition 12.18. $Ml(\check{\varphi}, \psi') = \bar{\varphi}\psi'$. a

This follows from 12.16.

19.15b

Proposition 12.19. $Ml(\check{\varphi}, I') = \bar{\varphi}$.

This follows from 12.18.

Proposition 12.20. $Ml(\check{\varphi}, \check{\psi}) = (\varphi\psi)\check{}$. c

This follows from 12.18, 12.15.

Proposition 12.21. $Ml(\varphi' Id, I') = \overline{\varphi'(I)}$. Whenever $\bar{\psi} Id = \varphi' Id$, then $\psi = \varphi'(I)$.
This follows from 12.14, 12.19.

Being especially interested in IOS, we assume from now on that both $\mathscr{S}$ and $\mathscr{S}'$ are iterative. The μ-axiom of $\mathscr{S}$ used will as usual be indicated by the corresponding number of asterisks. In order to ensure that $\bar{\mathscr{S}}$ is a subspace of $\mathscr{S}'$ isomorphic with $\mathscr{S}$ as an IOS, we assume finally that $\langle I' \rangle = \overline{\langle I \rangle}$.

Notice in connection with the last equality that $(L', \overline{\langle I \rangle} R') = \overline{(L, \langle I \rangle R)} = \overline{\langle I \rangle}$ implies $\langle I' \rangle \le \overline{\langle I \rangle}$. The converse inequality can also be proved by μA_2: $R'I' = I'R'$, $(L', \langle I' \rangle R') = \langle I' \rangle$ imply $\overline{\langle I \rangle} \le \langle I' \rangle$ by the suggested hint to exercise 12.1. Moreover, $\langle I' \rangle = \langle I' \rangle \overline{\langle I \rangle}$ by 6.22; hence $\langle I' \rangle = \overline{\langle I \rangle}$ whenever $\langle I' \rangle = I'$.

An immediate example of consecutive IOS is that of the IOS $\mathscr{S}$ of example 3.1 and the space $\mathscr{S}'$ obtained from it by 12.1. The latter is iterative since being isomorphic with the IOS of example 3.2, while the equality $\overline{\langle I \rangle} = \langle I' \rangle = I'$ also holds in this case.

Proposition 12.22. $\langle \bar{\varphi} \rangle = \overline{\langle \varphi \rangle}$.
Proof.

$$(\bar{\varphi} L', \overline{\langle \varphi \rangle} R') = \overline{(\varphi L, \langle \varphi \rangle R)} = \overline{\langle \varphi \rangle}$$

implies $\langle \bar{\varphi} \rangle = \langle I' \rangle \overline{\langle \varphi \rangle}$ by 6.22. Therefore,

$$\langle \bar{\varphi} \rangle = \overline{\langle I \rangle}\,\overline{\langle \varphi \rangle} = \overline{\langle I \rangle \langle \varphi \rangle} = \overline{\langle \varphi \rangle}.$$

Proposition 12.23. $[\bar{\varphi}] = \overline{[\varphi]}$.
Proof. We have

$$(I', \bar{\varphi}\overline{[\varphi]}) = \overline{(I, \varphi[\varphi])} = \overline{[\varphi]},$$

hence $[\bar{\varphi}] \le \overline{[\varphi]}$. On the other hand,

$$(\theta, \varphi[\bar{\varphi}](\theta)) = (I', \bar{\varphi}[\bar{\varphi}])(\theta) = [\bar{\varphi}](\theta)$$

implies $[\varphi]\theta \le [\bar{\varphi}](\theta)$ by (££). Therefore, $\overline{[\varphi]}(\theta) \le [\bar{\varphi}](\theta)$ for all θ, hence $\overline{[\varphi]} \le [\bar{\varphi}]$. The proof is complete.

14.19 **Proposition 12.24** (Imbedding Theorem). $\bar{\mathscr{S}}$ is an iterative subspace of $\mathscr{S}'$ isomorphic with the IOS $\mathscr{S}$.
This follows from 12.11, 12.22, 12.23.

20a **Proposition 12.25.** $[\varphi'] = \lambda\theta.\mu\theta_1.(\theta, \varphi'(\theta_1))$.
Proof. We have

$$[\varphi'](\theta) = (I', \varphi'[\varphi'])(\theta) = (\theta, \varphi'([\varphi'](\theta))).$$

Suppose that $(\theta, \varphi'(\tau)) \le \tau$. Then $(\check{\theta}, \varphi'\check{\tau}) \le \check{\tau}$, hence $[\varphi']\check{\theta} \le \check{\tau}$ by (££), i.e. $[\varphi'](\theta) \le \tau$. Therefore, $[\varphi'](\theta) = \mu\theta_1.(\theta, \varphi'(\theta_1))$ for all θ, which completes the proof. (Compare with the proof of 3.4.)

Proposition 12.26. $R'[\varphi'] = \lambda\theta.\mu\theta_1.\varphi'((\theta,\theta_1))$.

The proof follows that of 12.25, making use of 6.11 instead of (££).

Proposition 12.27. $R'[\varphi' R'] = (\mu\theta.\varphi'(\theta))\check{}$.

This follows from 12.26.

Proposition 12.28. Let $Tr = \langle\quad\rangle$. Then Tr is a member of $\mathscr{F}'$ prime recursive in Id, Ml.

Proof. Take $\sigma' = (Ml(L', \check{L}), Ml(R', \check{R}))$. Then $\sigma'((\theta,\theta_1)) = (\theta L, \theta_1 R)$ for all θ, θ_1, hence

$$Tr = \lambda\theta.\mu\theta_1.(\theta L, \theta_1 R) = R'[\sigma']$$

by 12.26.

Proposition 12.29. Let $It = [\quad]$. Then It is a member of $\mathscr{F}'$ prime recursive in Id, Ml.

Proof. Taking $\sigma' = (Id, Ml)$, we get, using 12.26

$$It = \lambda\theta.\mu\theta_1.(I, \theta\theta_1) = R'[\sigma'].$$

Proposition 12.30. Tr, $It \notin \bar{\mathscr{F}}$.

Proof. Suppose $Tr = \bar{\varphi}$. Then $\varphi = \bar{\varphi}(I) = Tr(I) = \langle I\rangle$, hence $\langle I\rangle L = \bar{\varphi}(L) = \langle L\rangle$, which is not the case. Iteration is treated similarly.

Proposition 12.31. If φ' is recursive, then it is prime recursive in Id, Ml.

Proof. All the recursive members of $\mathscr{F}'$ are in $\bar{\mathscr{F}}$. An easy induction on the construction of such an element $\bar{\varphi}$ shows that it is prime recursive in Id, Ml: we use of the equalities $\langle\bar{\varphi}\rangle = Ml(Tr\bar{\varphi}Id, I')$, $[\bar{\varphi}] = Ml(It\bar{\varphi}Id, I')$ which follow from 12.21 and take 12.28, 12.29 into account.

Proposition 12.32. $\langle\check{\varphi}\rangle = [I']\check{\varphi}$. In particular, $\langle Id\rangle = [I']Id$

Proof. We have

$$(\check{\varphi}L', [I']\check{\varphi}R') = (\check{\varphi}, [I']\check{\varphi}) = [I']\check{\varphi};$$

hence $\langle\check{\varphi}\rangle \leq [I']\check{\varphi}$. On the other hand,

$$(\check{\varphi}, \langle\check{\varphi}\rangle O') = (\check{\varphi}L', \langle\check{\varphi}\rangle R')O' = \langle\check{\varphi}\rangle O'$$

implies $[I']\check{\varphi} \leq \langle\check{\varphi}\rangle O'$ by (££); hence $[I']\check{\varphi} \leq \langle\check{\varphi}\rangle I' = \langle\check{\varphi}\rangle$. The proof is complete.

Proposition 12.33. $\langle Ml\rangle = \langle L'\rangle D' Ml(Tr Ml(I', \check{R}), I')$.

Proof. Writing ρ' for $\langle L'\rangle D' Ml(Tr Ml(I', \check{R}), I')$, we have

$$\begin{aligned}\rho'(\theta) &= \langle L\rangle D\langle\theta R\rangle\theta = (L^2, \langle L\rangle R)D\langle\theta R\rangle\theta \\ &= (L^3\langle\theta R\rangle\theta, \langle L\rangle D\langle R\rangle R\langle\theta R\rangle\theta) \\ &= (L^2\theta RL\theta, \langle L\rangle D\langle R\theta R\rangle R\theta) = (Ml(L\theta), \rho'(R\theta))\end{aligned}$$

for all θ, i.e. $\rho' = (MlL', \rho' R')$, which implies $\langle Ml \rangle = \langle I' \rangle \rho' = \rho'$ by 6.22. The proof is complete.

The following Pull Back Theorem generalizes 12.31.

Proposition 12.34. Let $\mathscr{B} \subseteq \mathscr{F}$, $\mathscr{B}' \subseteq \mathscr{F}'$. Then the following are equivalent.

(1) φ' is recursive in $\{Id, Ml\} \cup \check{\mathscr{B}} \cup \mathscr{B}'$.

(2) φ' is prime recursive in $\{Id, Ml\} \cup \check{\mathscr{B}} \cup \langle \mathscr{B}' \rangle$.

(3) $\varphi' \in cl(\{L', A', Id, Ml\} \cup \check{\mathscr{B}} \cup \langle \mathscr{B}' \rangle / \circ, [\quad])$.

If all the members of $\langle \mathscr{B}' \rangle$ are prime recursive in $\{Id, Ml\} \cup \check{\mathscr{B}} \cup \mathscr{B}'$, then $\mathscr{B}'$ can be substituted for $\langle \mathscr{B}' \rangle$ in (2), (3).

Proof. The implication (1)$\Rightarrow$(2) follows from 7.11, 12.31–12.33 and 12.28, while the other implications are immediate.

It should be mentioned that $\{Id, Ml\} \cup \check{\mathscr{B}}$ can be replaced by $\{Id, Ml\} \cup \bar{\mathscr{B}}$ above, in view of 12.3, 12.19. In future this fact will be employed without further mention.

The notion of inductive mapping over $\mathscr{F}$ can be relativized by substituting $\{I, L, R\} \cup \mathscr{B}$ for $\{I, L, R\}$ in the first clause of the definition given in chapter 5. The following statement characterizes this notion of relative inductiveness in the terms of $\mathscr{S}'$.

Proposition 12.35. Let Γ be a $n+1$-ary mapping over $\mathscr{F}$ inductive in $\mathscr{B}$ and let Γ^* correspond to Γ by 7.20. Then Γ^* is a member of $\mathscr{F}'$ prime recursive in $\{Id, Ml\} \cup \check{\mathscr{B}}$. Moreover, $\mu\theta.\Gamma(\theta_1, \ldots, \theta_n, \theta)$ exists for all $\theta_1, \ldots, \theta_n$.

Proof. By induction on the construction of Γ.

Let $\Gamma = \lambda\theta_1 \ldots \theta_n.\theta_i$, $1 \leq i \leq n$. If $i < n$, then $\Gamma^* = \lambda\theta.\bar{i}\theta = L'R'^i$. If $i = n$, then $\Gamma^* = \lambda\theta.R^{n-1}\theta = R'^{n-1}$.

If $\Gamma = \lambda\theta_1 \ldots \theta_n.\psi$, $\psi \in \{I, L, R\} \cup \mathscr{B}$, then $\Gamma^* = \lambda\theta.\psi = \check{\psi}$.

Let Γ_1, Γ_2 be n-ary and Γ_1^*, $\Gamma_2^* \in \mathscr{F}'$ correspond to Γ_1, Γ_2.

If $\Gamma = \lambda\theta_1 \ldots \theta_n.\Gamma_1(\theta_1, \ldots, \theta_n)\Gamma_2(\theta_1, \ldots, \theta_n)$, then $\Gamma^* = \lambda\theta.\Gamma_1^*(\theta)\Gamma_2^*(\theta) = Ml(\Gamma_1^*, \Gamma_2^*)$.

If $\Gamma = \lambda\theta_1 \ldots \theta_n.(\Gamma_1(\theta_1, \ldots, \theta_n), \Gamma_2(\theta_1, \ldots, \theta_n))$, then $\Gamma^* = \lambda\theta.(\Gamma_1^*(\theta), \Gamma_2^*(\theta)) = (\Gamma_1^*, \Gamma_2^*)$.

Let Γ_1 be $n+1$-ary, $n > 0$, and $\Gamma_1^* \in \mathscr{F}'$ correspond to Γ_1. Take

$$\Gamma^* = R'[\Gamma_1^*(L'^2, \ldots, L'R'^{n-2}L', R'^{n-1}L', R')].$$

Then it follows that $\Gamma^* = \lambda\theta.\mu\theta_1.\Gamma_1^*((\bar{0}\theta, \ldots, \overline{n-2}\theta, R^{n-1}\theta, \theta_1))$ by 12.26, hence Γ^* corresponds to the maping $\Gamma = \lambda\theta_1 \ldots \theta_n.\mu\theta.\Gamma_1(\theta_1, \ldots, \theta_n, \theta)$. This completes the proof.

Proposition 12.36. Whenever $\varphi' \in \mathscr{F}'$ is prime recursive in $\{Id, Ml\} \cup \check{\mathscr{B}}$, then φ' is a mapping inductive in $\mathscr{B}$.

This follows from an easy induction on the construction of φ'. In particular, if φ' is inductive in $\mathscr{B}$, then so is $[\varphi']$ since $[\varphi'] = \lambda\theta.\mu\theta_1.(\theta, \varphi'(\theta_1))$ by 12.25.

Proposition 12.37. Let Γ be a unary mapping over $\mathscr{F}$. Then the following are equivalent.

(1) Γ is a mapping inductive in $\mathscr{B}$.

(2) Γ is a member of $\mathscr{F}'$ prime recursive in $\{Id, Ml\} \cup \check{\mathscr{B}}$.

(3) Γ is a member of $\mathscr{F}'$ recursive in $\{Id\ Ml\} \cup \check{\mathscr{B}}$.

This follows from 12.34–12.36.

The following Conservativeness Theorem is the central result of this chapter.

19.31*

Proposition 12.38*. Let Γ be a unary mapping over $\mathscr{F}$. Then the following are equivalent.

(1) Γ is a mapping recursive in $\mathscr{B}$.

(2) Γ is a member of $\mathscr{F}'$ prime recursive in $\{Id, Ml\} \cup \check{\mathscr{B}}$.

(3) Γ is a member of $\mathscr{F}'$ recursive in $\{Id, Ml\} \cup \check{\mathscr{B}}$.

This follows from 12.37, 9.9 and 9.15*. Notice that $\check{\mathscr{B}}$ can be replaced by $\bar{\mathscr{B}}$.

Therefore, the relative recursiveness of $\mathscr{S}$ is conservative in the next space $\mathscr{S}'$. In other words, the initial operations of $\mathscr{S}'$ preserve it. This is immediate for $\circ$, Π and quite apparent for $[\ \ ]$ since the iteration of $\mathscr{S}'$ is in essence the μ-operation over $\mathscr{F}$ by 12.25. However, the fact that the operation translation of $\mathscr{S}'$ preserves the relative recursiveness of $\mathscr{S}$ is curious and worth stating separately.

Proposition 12.39. If $\varphi' \in \mathscr{F}'$ is recursive in $\mathscr{B}$ as a mapping over $\mathscr{F}$, then so is $\langle \varphi' \rangle$.

Proof. Take Γ^* to correspond to φ' by the Transition Theorem. Then Γ^* is recursive in $\mathscr{B}$ and $\Gamma^*(\theta) = (\varphi'(L\theta), \Gamma^*(R\theta))$ for all θ. It follows from 12.35 that $\Gamma^* \in \mathscr{F}'$, hence $\Gamma^* = (\varphi' L', \Gamma^* R')$, which implies $\langle \varphi' \rangle = \langle I' \rangle \Gamma^*$ by 6.22. However, $\langle I \rangle \Gamma^*(\theta) = \Gamma^*(\theta)$ for all θ, hence $\langle I' \rangle \Gamma^* = \overline{\langle I \rangle} \Gamma^* = \Gamma^*$ and $\langle \varphi' \rangle = \Gamma^*$. The proof is complete.

Exercise 7.6 and the above proof imply that the operation translation of $\mathscr{S}'$ also preserves the relative primitive recursiveness of $\mathscr{S}$. Therefore, propositions 7.21, 12.39, exercise 7.6 and proposition 12.25 throw some more light on the nature of the operations translation and iteration.

Assume now that $\langle\!\!\langle\ \ \rangle\!\!\rangle$ is a t-operation over $\mathscr{F}$ with a corresponding set of functional elements $\mathscr{B}_0$ such that $To = \langle\!\!\langle\ \ \rangle\!\!\rangle \in \mathscr{F}'$. Then one may establish analogues to the above statements by adding $\check{\mathscr{B}}_0$ and To to the initial elements Id, Ml of $\mathscr{F}'$. Thus let us write $\mathscr{B}_0'$ for $\{Id, Ml, To\} \cup \check{\mathscr{B}}_0$.

Proposition 12.40. The element $\langle To \rangle$ is prime recursive in $\mathscr{B}_0'$.

Proof. Take the mappings $\langle\!\!\langle\ \ \rangle\!\!\rangle^*, \Gamma_1^*, \Gamma_3$ considered in the proof of 10.7. The mapping Γ_1^*, Γ_3 are recursive in $\mathscr{B}_0$, hence they are members of $\mathscr{F}'$ prime recursive in $\{Id, Ml\} \cup \check{\mathscr{B}}_0$ by 12.35. Therefore, $\langle\!\!\langle\ \ \rangle\!\!\rangle^*$ is a member of $\mathscr{F}'$ prime recursive in $\mathscr{B}_0'$ since $\langle\!\!\langle\ \ \rangle\!\!\rangle^* = \Gamma_1^* \Gamma_3 To$. It follows that $\langle To \rangle = \langle\!\!\langle\ \ \rangle\!\!\rangle^*$ by 6.22. The proof is complete.

10.7 guarantees that the operation $\langle\ \ \rangle$ of $\mathscr{S}'$ preserves the relative t-recursiveness of $\mathscr{S}$.

Making use of 12.40 and the proof of 12.34, one gets the following analogue to the latter statement.

Proposition 12.41. Let $\mathscr{B} \subseteq \mathscr{F}$, $\mathscr{B}' \subseteq \mathscr{F}'$. Then the following are equivalent.

(1) φ' is recursive in $\mathscr{B}'_0 \cup \check{\mathscr{B}} \cup \mathscr{B}'$.

(2) φ' is prime recursive in $\mathscr{B}'_0 \cup \check{\mathscr{B}} \cup \langle \mathscr{B}' \rangle$.

(3) $\varphi' \in cl(\{A'\} \cup \mathscr{B}'_0 \cup \check{\mathscr{B}} \cup \langle \mathscr{B}' \rangle / \circ, [\ \ ])$.

If all the member of $\langle \mathscr{B}' \rangle$ are prime recursive in $\mathscr{B}'_0 \cup \check{\mathscr{B}} \cup \mathscr{B}'$, then $\mathscr{B}'$ can be substituted for $\langle \mathscr{B}' \rangle$ in (2), (3).

Finally, the Parametrized First t-Recursion Theorem and an analogue to 12.37 give the following t-Conservativeness Theorem.

Proposition 12.42*. Let Γ be a unary mapping over $\mathscr{F}$. Then the following are equivalent.

(1) Γ is a mapping t-recursive in $\mathscr{B}$.

(2) Γ is a member of $\mathscr{F}'$ prime recursive in $\mathscr{B}'_0 \cup \check{\mathscr{B}}$.

(3) Γ is a member of $\mathscr{F}'$ recursive in $\mathscr{B}'_0 \cup \check{\mathscr{B}}$.

EXERCISES TO CHAPTER 12

Exercise 12.1. Let $\mathscr{S}$, $\mathscr{S}'$ be consecutive OS, and suppose that $\mathscr{S}$ is μA_2-iterative and the equality $(\varphi' L', \theta' R') = \theta'$ has a solution in $\mathscr{F}'$ for all φ'. Show that $\mathscr{S}'$ satisfies (£).

Hint. It suffices by 5.11 to find an element $\langle I' \rangle$ satisfying (£). Take $\langle I' \rangle = \overline{\langle I \rangle}$ and show that $(L', \langle I' \rangle R') = \langle I' \rangle$. Supposing $R'\psi' \leq \psi'\psi'_1$ and $(L'\psi', \tau'\psi'_1) \leq \tau'$, show that for all $\rho \in \mathscr{F}$ the normal segment $\mathscr{E} = \{\theta / \forall n(\theta\psi'\psi_1'^n(\rho) \leq \tau'\psi_1'^n(\rho))\}$ is closed under $\lambda\theta.(L, \theta R)$.

Exercise 12.2. Let $\mathscr{S}$, $\mathscr{S}'$ be consecutive OS and suppose that $\mathscr{F}'$ is closed under μ-operation over $\mathscr{F}$, i.e. whenever $\varphi' \in \mathscr{F}'$, then $\mu\theta_1.(\theta, \varphi'(\theta_1))$ exists for all θ and $\lambda\theta.\mu\theta_1.(\theta, \varphi'(\theta_1)) \in \mathscr{F}'$. Show that $\mathscr{S}'$ meets (££).

Hint. Take $[\varphi'] = \lambda\theta.\mu\theta_1.(\theta, \varphi'(\theta_1))$ and show that $(I', \varphi'[\varphi']) = [\varphi']$. If $(\psi', \varphi'\tau') \leq \tau'$, then $(\psi'(\theta), \varphi'(\tau'(\theta))) \leq \tau'(\theta)$; hence $[\varphi'](\psi'(\theta)) \leq \tau'(\theta)$ for all θ, i.e. $[\varphi']\psi' \leq \tau'$.

By 12.25, the requirement of the last exercise are necessary and sufficient for (££).

Assume now that $\mathscr{S}$, $\mathscr{S}'$ are consecutive IOS. Let $\Gamma_1, \ldots, \Gamma_m$ be unary mappings over $\mathscr{F}$ recursive in $\mathscr{B}$ and $\mathscr{V}(\theta, \Gamma)$ be the expression considered in exercise 9.8. Then $\Gamma_1, \ldots, \Gamma_m$ are members of $\mathscr{F}'$ recursive in $\{Id, Ml\} \cup \check{\mathscr{B}}$ by 12.37 and it follows that there is a unary mapping Γ' over $\mathscr{F}'$ polynomial in $\Gamma_1, \ldots, \Gamma_m$ such that $\Gamma'(\Gamma) = \lambda\theta.\mathscr{V}(\theta, \Gamma)$ for all $\Gamma \in \mathscr{F}'$. The First Recursion Theorem for $\mathscr{S}'$ implies that Γ' has a least fixed point Γ recursive in $\Gamma_1, \ldots, \Gamma_m$, hence recursive in $\{Id, Ml\} \cup \check{\mathscr{B}}$. (Proposition 9.11 may also serve, provided $c(\Gamma') \leq 1$.) Therefore, Γ is recursive in $\mathscr{B}$ by 12.38*,

$$\mathscr{V}(\theta, \Gamma) = \Gamma(\theta), \text{ all } \theta, \tag{1}$$

and $\Gamma \leq \Gamma^*$ whenever $\Gamma^* \in \mathscr{F}'$ meets (1). However, (1) may have solutions which are nonmonotonic mappings and hence not in $\mathscr{F}'$. The next two exercises will show that Γ is least among all solutions of (1).

Take $\mathscr{F}'' = \{\varphi''/\varphi'':\mathscr{F} \to \mathscr{F}\}$ and introduce $\leq$, $\circ$, Π'' as in 12.1. Then $\mathscr{S}'' = (\mathscr{F}'', I', \Pi'', L', R')$ is almost an OS except for the nonmonotonicity of $\circ$ on its second argument. Notice that the OS $\mathscr{S}'$ is a 'subspace' of $\mathscr{S}''$ and whenever $\psi'' \leq \psi_1''$, then $\varphi'\psi'' \leq \varphi'\psi_1''$ for all $\varphi' \in \mathscr{F}'$.

Exercise 12.3. Let $\mathscr{S}'$ be μA_2-iterative and Γ' a unary mapping over $\mathscr{F}'$ polynomial in $\mathscr{F}'$. Show that $\mu\theta'.\Gamma'(\theta') = \mu\theta''.\Gamma'(\theta'')$ where the domain of Γ' is naturally extended to $\mathscr{F}''$.

Hint. Show first by induction on the construction of Γ' that $\theta' \leq \tau''$ implies $\Gamma'(\theta') \leq \Gamma'(\tau'')$. Assuming $\Gamma'(\tau'') \leq \tau''$, show that the normal segment

$$\mathscr{E}' = \{\theta'/\theta' \leq \tau''\} = \{\theta'/\forall\theta(\theta'\check{\theta} \leq (\tau''(\theta))\check{})\}$$

is closed under Γ'; hence $\mu\theta'.\Gamma'(\theta') \in \mathscr{E}'$ by μA_2.

While the above set of inequalities defining $\mathscr{E}'$ is most probably uncountable, its role could be played by countable one reflecting the construction of Γ'. A simple example outlines the idea. Let $\Gamma' = \lambda\theta'.\varphi'(I', \theta'\psi'\theta')$ and $\Gamma'(\tau'') \leq \tau''$. Then fix $\theta \in \mathscr{F}$ and show that the normal segment

$$\mathscr{E}' = \{\theta'/\forall n(\theta'(\psi'\tau'')^n\check{\theta} \leq \tau''(\psi'\tau'')^n\check{\theta})\}$$

is closed under Γ'.

The following exercise establishes a Generalized First Recursion Theorem for $\mathscr{S}$.

Exercise 12.4*. Let $\mathscr{S}'$ be μA_2-iterative, let $\mathscr{V}(\theta, \Gamma)$ be the expression considered in exercise 9.8 and suppose that $\Gamma_1, \ldots, \Gamma_m$ are recursive in $\mathscr{B}$. Show that there is a unary mapping Γ recursive in $\mathscr{B}$ such that $\mathscr{V}(\theta,\Gamma) = \Gamma(\theta)$ for all θ and whenever $\Gamma^*:\mathscr{F} \to \mathscr{F}$ and $\mathscr{V}(\theta, \Gamma^*) \leq \Gamma^*(\theta)$ for all θ, then $\Gamma(\theta) \leq \Gamma^*(\theta)$ for all θ.

Hint. Use 9.13* for $\mathscr{S}'$, exercise 12.3 and 12.38*.

The following exercise establishes a Generalized First t-Recursion Theorem.

Exercise 12.5*. Let $\mathscr{S}'$ be μA_2-iterative, let $\langle\!\!\!\!\langle\ \ \rangle\!\!\!\!\rangle \in \mathscr{F}'$ be a t-operation over $\mathscr{F}$, let $\mathscr{V}(\theta, \Gamma)$ be the expression of exercise 9.8 and suppose that $\Gamma_1, \ldots, \Gamma_m$ are t-recursive in $\mathscr{B}$. Show that there is a unary mapping Γ t-recursive in $\mathscr{B}$ such that $\mathscr{V}(\theta, \Gamma) = \Gamma(\theta)$ for all θ and, if $\Gamma^*:\mathscr{F} \to \mathscr{F}$ and $\mathscr{V}(\theta, \Gamma^*) \leq \Gamma^*(\theta)$ for all θ, then $\Gamma(\theta) \leq \Gamma^*(\theta)$ for all θ.

Hint. Follow the hint to the previous exercise, using 12.42* instead of 12.38*.

As was the case for exercise 9.8, exercises 12.4*, 12.5* can be restated for n-ary mappings as well.

Exercise 12.6. Show that φ' is recursive in $\{\check{\psi}_1, \ldots, \check{\psi}_m\} \cup \mathscr{B}'$ iff there is a χ recursive in $\mathscr{B}'$ such that $\varphi' = \chi'(\check{\psi}_1, \ldots, \check{\psi}_m, I')$.

Hint. Make use of 12.4, 12.32, 6.11.

CHAPTER 13

$\mathscr{B}'$-Recursiveness

Let us recall that the concept of consecutive spaces was introduced in chapter 12 because we wanted to study monotonic operations other than the initial IOS-operations. So take consecutive IOS $\mathscr{S}$, $\mathscr{S}'$ and fix a subset $\mathscr{B}'$ of $\mathscr{F}'$. We are interested in elements of $\mathscr{F}$ and mappings over $\mathscr{F}$ which can be constructed by making use of operations from $\mathscr{B}'$. This leads to the following definitions.

A unary mapping Γ and $\mathscr{F}$ is said to be *$\mathscr{B}'$-recursive in* $\mathscr{B} \subseteq \mathscr{F}$ iff Γ is a member of $\mathscr{F}'$ recursive in $\{Id, Ml\} \cup \check{\mathscr{B}} \cup \mathscr{B}'$. An element φ is *$\mathscr{B}'$-recursive in* $\mathscr{B}$ iff $\check{\varphi}$ is a mapping $\mathscr{B}'$-recursive in $\mathscr{B}$. Therefore, the class $\mathscr{R}(\mathscr{B}, \mathscr{B}')$ of unary mappings $\mathscr{B}'$-recursive in $\mathscr{B}$ equals

$$cl(\check{\mathscr{B}} \cup \{L', R', Id, Ml\} \cup \mathscr{B}'/\circ, \Pi, \langle\quad\rangle, [\quad]).$$

While the initial operations of $\mathscr{S}'$ are allowed by the above definition, 12.16, 12.28, 12.29 imply that those of $\mathscr{S}$ are also available. In some favourable cases the operations $\langle\quad\rangle$, $[\quad]$ of $\mathscr{S}'$ can be eliminated and one may consider the class

$$\mathscr{R}_0(\mathscr{B}, \mathscr{B}') = cl(\check{\mathscr{B}} \cup \{L', R', Id, Ml, Tr, It\} \cup \mathscr{B}'/\circ, \Pi)$$

instead. For instance, $\mathscr{R}_0(\mathscr{B}, \varnothing) = \mathscr{R}(\mathscr{B}, \varnothing)$ by 12.38*.

It is obvious that $\mathscr{R}_0(\mathscr{B}, \mathscr{B}') \subseteq \mathscr{R}(\mathscr{B}, \mathscr{B}')$. In order to get equality, one should prove first that all the mappings in $\mathscr{R}_0(\mathscr{B}, \mathscr{B}')$ satisfy the transition property, thus ensuring that $\mathscr{R}_0(\mathscr{B}, \mathscr{B}')$ is closed under the operation $\langle\quad\rangle$ of $\mathscr{S}'$. This is not a major obstacle since translation can be pulled back to the members of $\mathscr{B}'$ by 12.34 anyway. Secondly, one should prove a Parametrized First Recursion Theorem to ensure that $\mathscr{R}_0(\mathscr{B}, \mathscr{B}')$ is closed under μ-operation over $\mathscr{F}$, hence closed under the operation $[\quad]$ of $\mathscr{S}'$. This cannot be expected to be the case for arbitrary subsets $\mathscr{B}'$ of $\mathscr{F}'$ and as a result $\mathscr{R}_0(\mathscr{B}, \mathscr{B}')$ may differ from $\mathscr{R}(\mathscr{B}, \mathscr{B}')$.

Now if $\mathscr{B}'$ is *finite*, then there is by 9.18 a σ' recursive in $\{Id, Ml\} \cup \mathscr{B}'$ and universal for all the members of $\mathscr{F}'$ recursive in $\{Id, Ml\} \cup \mathscr{B}'$. Therefore, $\mathscr{R}(\mathscr{B}, \mathscr{B}') = cl(\check{\mathscr{B}} \cup \{I', L', R', \sigma'\}/\circ, \Pi)$ by exercise 12.6, hence $\mathscr{R}_0(\mathscr{B}, \{\sigma'\}) = \mathscr{R}(\mathscr{B}, \{\sigma'\})$ for all $\mathscr{B}$. However, while the replacement of $\mathscr{B}'$ by $\langle \mathscr{B}' \rangle$ can still be tolerated, the replacement of $\mathscr{B}'$ by σ' cannot. The mapping σ' can hardly be accepted as an initial operation, being far too sophisticated.

The definition of $\mathscr{B}'$-recursiveness assumes a First Recursion Theorem at the very beginning; it would be preferable to prove such a property rather than include it in the definition. Therefore, our ability to treat arbitrary monotonic operations over $\mathscr{F}$ is achieved only at a price.

Notions of mappings and elements *prime $\mathscr{B}'$-recursive* (respectively, *primitive $\mathscr{B}'$-recursive, $\mathscr{B}'$-primitive, $\mathscr{B}'$-polynomial) in $\mathscr{B}$* are introduced in the same way. For example, a unary mapping Γ is prime $\mathscr{B}$-recursive in $\mathscr{B}$ iff it is a member of $\mathscr{F}'$ prime recursive in $\{Id, Ml\} \cup \breve{\mathscr{B}} \cup \mathscr{B}'$, an element φ is prime $\mathscr{B}'$-recursive in $\mathscr{B}$ iff so is the mapping $\breve{\varphi}$. Notice that $\bar{\varphi}$, $\bar{\mathscr{B}}$ can be substituted for $\breve{\varphi}$, $\breve{\mathscr{B}}$ by 12.3, 12.19. Further on $\bar{n}$ will stand for LR^n or $L'R'^n$, depending on the context.

The notions thus introduced have certain ordinary properties similar to those stated at the beginning of chapter 7. Some of them are listed below, mostly concerning mappings. Obvious proofs are omitted.

Proposition 13.1. If $\varphi \in \mathscr{B}$, then φ is $\mathscr{B}'$-polynomial in $\mathscr{B}$. If $\Gamma \in \mathscr{B}'$, then Γ is $\mathscr{B}'$-polynomial in $\mathscr{B}$.

Proposition 13.2. If Γ is $\mathscr{B}'$-polynomial in $\mathscr{B}$, then it is both $\mathscr{B}'$-primitive in $\mathscr{B}$ and prime $\mathscr{B}'$-recursive in $\mathscr{B}$. If Γ is $\mathscr{B}'$-primitive in $\mathscr{B}$, then it is primitive $\mathscr{B}'$-recursive in $\mathscr{B}$. If Γ is prime or primitive $\mathscr{B}'$-recursive in $\mathscr{B}$, then it is $\mathscr{B}'$-recursive in $\mathscr{B}$.

Proposition 13.3. If $\mathscr{B} \subseteq \mathscr{B}_1$, $\mathscr{B}' \subseteq \mathscr{B}'_1$ and Γ is $\mathscr{B}'$-recursive (prime $\mathscr{B}'$-recursive etc.) in $\mathscr{B}$, then Γ is $\mathscr{B}'_1$-recursive (prime $\mathscr{B}'_1$-recursive etc.) in $\mathscr{B}_1$.

Proposition 13.4. If Γ is $\mathscr{B}'$-recursive (prime $\mathscr{B}'$-recursive etc.) in $\mathscr{B}$ and all the members of $\mathscr{B}$ are $\mathscr{B}'$-recursive (prime $\mathscr{B}'$-recursive etc.) in $\mathscr{B}_1$, then so is Γ.

Proposition 13.5. If Γ is $\mathscr{B}'$-recursive (prime $\mathscr{B}'$-recursive etc.) in $\mathscr{B}$ and all the members of $\mathscr{B}'$ are $\mathscr{B}'_1$-recursive (prime $\mathscr{B}'_1$-recursive etc.) in $\mathscr{B}$, then so is Γ.

Proposition 13.6. If Γ is $\mathscr{B}'$-recursive (prime $\mathscr{B}'$-recursive etc.) in $\mathscr{B}$, then there are finite subsets $\mathscr{B}_1$, $\mathscr{B}'_1$ of $\mathscr{B}$, $\mathscr{B}'$ respectively, such that Γ is $\mathscr{B}'_1$-recursive (prime $\mathscr{B}'_1$-recursive etc.) in $\mathscr{B}_1$.

Proposition 13.7. A mapping Γ is $\mathscr{B}'$-recursive in $\psi_1, \ldots, \psi_m$ iff there is a $\mathscr{B}'$-recursive mapping Γ^* such that $\Gamma = \lambda\theta.\Gamma^*((\psi_1, \ldots, \psi_m, \theta))$.

This follows from exercise 12.6.

Proposition 13.8. Γ is $\mathscr{B}'$-recursive in $\mathscr{B}$ iff Γ is prime $\langle \mathscr{B}' \rangle$-recursive in $\mathscr{B}$.

This follows from 12.34.

Proposition 13.9*. Γ is $\varnothing$-recursive in $\mathscr{B}$ iff Γ is recursive in $\mathscr{B}$.

This follows from 12.38*.

It is also the case that Γ is $\varnothing$-polynomial in $\mathscr{B}$ iff Γ is polynomial in $\mathscr{B}$. However, such an equivalence may fail for the other notions. Indeed, the mappings prime $\varnothing$-recursive in $\mathscr{B}$ are those recursive in $\mathscr{B}$ by 12.38*, while the mappings prime recursive in $\mathscr{B}$ are those $\{It\}$-polynomial in $\mathscr{B}$.

We next prove several Normal Form Theorems for prime $\mathscr{B}'$-recursiveness and $\mathscr{B}'$-recursiveness.

Proposition 13.10. If Γ is prime $\mathscr{B}'$-recursive in $\mathscr{B}$, then $\Gamma = \lambda\theta.L\mu\theta_1.\Gamma_0((\theta,\theta_1))$ for a certain Γ_0 strictly $\mathscr{B}'$-polynomial in $\mathscr{B}$. (That is, Γ_0 is a member of $\mathscr{F}'$ strictly polynomial in $\check{\mathscr{B}} \cup \{Id, Ml\} \cup \mathscr{B}'$.)

Proof. It follows from 9.2 that $\Gamma = \bar{1}[\Gamma_0]$ with Γ_0 strictly $\mathscr{B}'$-polynomial in $\mathscr{B}$. Therefore, Γ has the desired normal form by 12.26.

Proposition 13.11. If φ is prime $\mathscr{B}'$-recursive in $\mathscr{B}$, then $\varphi = L\mu\theta.\Gamma(\theta)$ for a certain Γ strictly $\mathscr{B}'$-polynomial in $\mathscr{B}$.

Proof. This proof needs some care since a direct application of 13.10 to $\check{\varphi}$ would only give a normal form with Γ $\mathscr{B}'$-polynomial in $\mathscr{B}$.

Proposition 9.2 implies that $\check{\varphi} = \bar{1}[\Gamma_0]$ with Γ_0 strictly $\mathscr{B}'$-polynomial in $\mathscr{B}$. Multiplying by Id and using 6.13, 6.14, we get

$$\check{\varphi} = \bar{2}[(Id, \Gamma_0 R')] = \bar{1}[(\bar{4}, \bar{0}, Id, \Gamma_0 R'^3)],$$

hence $\check{\varphi} = \lambda\theta.L\mu\theta_1.(\bar{3}\theta_1, \theta, I, \Gamma_0 R'^2(\theta_1))$ by 12.26. The mapping $\Gamma = (\bar{3}, Id, Id, \Gamma_0 R'^2)$ is strictly $\mathscr{B}'$-polynomial in $\mathscr{B}$ and $\varphi = \check{\varphi}(I) = L\mu\theta.\Gamma(\theta)$. This completes the proof.

Proposition 13.12. If Γ is $\mathscr{B}'$-recursive in $\mathscr{B}$, then $\Gamma = \lambda\theta.L\mu\theta_1.\Gamma_0((\theta,\theta_1))$ for a certain Γ_0 strictly $\langle\mathscr{B}'\rangle$-polynomial in $\mathscr{B}$.

This follows from 13.8, 13.10. Of course, Γ_0 is $\mathscr{B}'$-primitive in $\mathscr{B}$.

Proposition 13.13. If φ is $\mathscr{B}'$-recursive in $\mathscr{B}$, then $\varphi = L\mu\theta.\Gamma(\theta)$ with Γ strictly $\langle\mathscr{B}'\rangle$-polynomial in $\mathscr{B}$.

This follows from 13.8, 13.11.

Further normal form results are provided by the Normal Form Theorem 9.5 for $\mathscr{S}'$. Take $\mathscr{B}_0$ as in chapter 9. Its role in $\mathscr{S}'$ can be played by $\bar{\mathscr{B}}_0$, so the strictly primitive members of $\mathscr{F}'$ will be those of the form $\bar{\varphi}$ with φ a strictly primitive member of $\mathscr{F}$.

Proposition 13.14. If Γ is $\mathscr{B}'$-recursive in $\mathscr{B}$, then

$$\Gamma = \lambda\theta.L\mu\theta_1.\psi(\psi_0,\dots,\psi_m,\theta,\theta_1,\langle\Gamma_0\rangle(\theta_1),\dots,\langle\Gamma_n\rangle(\theta_1))$$

for certain $\psi_0,\dots,\psi_m \in \{I\} \cup \mathscr{B}, \Gamma_0,\dots,\Gamma_n \in \{Ml\} \cup \mathscr{B}'$ and a strictly primitive ψ.

Proof. Proposition 9.5 implies that

$$\Gamma = \bar{1}[\bar{\chi}(I', \langle\check{\psi}_0\rangle,\dots,\langle\check{\psi}_m\rangle,\langle\Gamma_0\rangle,\dots,\langle\Gamma_n\rangle)]$$

with $\psi_0,\dots,\psi_m \in \{I\} \cup \mathscr{B}$, $\Gamma_0,\dots,\Gamma_n \in \{Ml\} \cup \mathscr{B}'$ and a recursive χ. Substituting $\overline{[I]}\check{\psi}_i$ for $\langle\check{\psi}_i\rangle$ by 12.32, we get

$$\Gamma = \bar{1}[\bar{\chi}_1(I', \check{\psi}_0,\dots,\check{\psi}_m,\langle\Gamma_0\rangle,\dots,\langle\Gamma_n\rangle)]$$

for a recursive χ_1. Repeating the last part of the proof of 9.4 and making use of 12.4, we get

$$\Gamma = \bar{1}[\bar{\psi}(\breve{\psi}_0, \ldots, \breve{\psi}_m, L', R', \langle \Gamma_0 \rangle R', \ldots, \langle \Gamma_n \rangle R')]$$

with a strictly primitive ψ, which completes the proof by 12.26.

Proposition 13.15. If φ' is $\mathscr{B}'$-recursive in $\mathscr{B}$, then

$$\varphi = \bar{1}\mu\theta.\psi(\psi_0, \ldots, \psi_m, \theta, \langle \Gamma_0 \rangle(\theta), \ldots, \langle \Gamma_n \rangle(\theta))$$

for certain $\psi_0, \ldots, \psi_m \in \{I\} \cup \mathscr{B}, \Gamma_0, \ldots, \Gamma_n \in \{Ml\} \cup \mathscr{B}'$ and a strictly primitive ψ.

Proof. Following the previous proof, we get

$$\breve{\varphi} = \bar{1}[\bar{\chi}(\breve{\psi}_0, \ldots, \breve{\psi}_m, I', \langle \Gamma_0 \rangle, \ldots, \langle \Gamma_n \rangle)]$$

for a strictly primitive χ. Multiplying by $\breve{\psi}_0$ and taking $\psi = (L, \chi)$, we get by 6.13

$$\breve{\varphi} = \bar{2}[\bar{\psi}(\breve{\psi}_0, \ldots, \breve{\psi}_m, R', \langle \Gamma_0 \rangle R', \ldots, \langle \Gamma_n \rangle R')],$$

which completes the proof by 12.27.

One may assume without any loss of generality that $\psi_0 = I$, $\Gamma_0 = Ml$ and $\psi_i \neq \psi_j$, $\Gamma_i \neq \Gamma_j$ for $i \neq j$. It should also be mentioned that the above normal form results do not imply those of chapter 9 by taking $\mathscr{B}' = \varnothing$.

The following two statements are called respectively First $\mathscr{B}'$-Recursion Theorem and Parametrized First $\mathscr{B}'$-Recursion Theorem. In contrast to the corresponding theorems of chapter 9 their proofs are quite trivial and need no μ-axiom stronger then (££).

Proposition 13.16. If Γ is a mapping $\mathscr{B}'$-recursive in $\mathscr{B}$, then $\mu\theta.\Gamma(\theta)$ exists and is an element $\mathscr{B}'$-recursive in $\mathscr{B}$.

This follows from 12.27.

Proposition 13.17. If Γ is a mapping $\mathscr{B}'$-recursive in $\mathscr{B}$, then $\lambda\theta.\mu\theta_1.(\theta, \Gamma(\theta_1))$ exists and is a mapping $\mathscr{B}'$-recursive in $\mathscr{B}$.

This follows from 12.25.

The following statements are called respectively Parametrized $\mathscr{B}'$-Enumeration Theorem and $\mathscr{B}'$-Enumeration Theorem.

Proposition 13.18. Let $\mathscr{B}$, $\mathscr{B}'$ be finite and $\mathscr{M} = \mathscr{R}(\mathscr{B}, \mathscr{B}')$. Then there is a mapping $\Sigma \in \mathscr{M}$ universal for $\mathscr{M}$.

Proof. $\mathscr{M}$ is the set of all the members of $\mathscr{F}'$ recursive in $\breve{\mathscr{B}} \cup \{Id, Ml\} \cup \mathscr{B}'$, hence there is a $\Sigma \in \mathscr{M}$ universal for $\mathscr{M}$ by 9.18. If $\Gamma \in \mathscr{M}$, then there is a n such that $\Gamma = \bar{n}\Sigma$ in $\mathscr{S}'$, hence $\Gamma(\theta) = \bar{n}\Sigma(\theta)$ for all θ. The proof is complete.

By 13.7, it follows that whenever $\mathscr{B}'$ is finite, then there is a $\mathscr{B}'$-recursive mapping Σ such that $\Gamma = \lambda\theta.\bar{n}\Sigma((\psi_1, \ldots, \psi_m, \theta))$ for a certain n, provided that Γ is $\mathscr{B}'$-recursive in $\psi_1, \ldots, \psi_m$.

Proposition 13.19. Let $\mathscr{B}$, $\mathscr{B}'$ be finite and $\mathscr{U} = \{\varphi / \breve{\varphi} \in \mathscr{M}\}$. Then there is a $\sigma \in \mathscr{U}$ which is universal for $\mathscr{U}$.

Proof. Take $\sigma = \Sigma(I)$, where Σ is the mapping considered in the previous proof. It follows that $\sigma \in \mathscr{U}$ since $\check{\sigma} = \Sigma Id$. If $\varphi \in \mathscr{U}$, then $\check{\varphi} \in \mathscr{M}$, hence $\check{\varphi} = \bar{n}\Sigma$ for a certain n. Therefore, $\check{\varphi} = \bar{n}\Sigma Id = \bar{n}\check{\sigma}$ which implies $\varphi = \bar{n}\sigma$. The proof is complete.

The notion of element (respectively mapping) principal universal for $\mathscr{U}$ (for $\mathscr{M}$) is introduced exactly as in chapter 9. All the statements of chapter 9 concerning such elements and mappings remain valid with '$\mathscr{B}'$-recursive in $\mathscr{B}$' substituted for 'recursive in $\mathscr{B}$'. It is worth mentioning that these considerations on universal elements and mappings make use of the operation $\langle\ \ \rangle$ of $\mathscr{S}'$ since so do the proofs of 13.18, 13.19, while propositions 13.16, 13.17 hold for prime $\mathscr{B}'$-recursiveness as well.

If $\langle\!\langle\ \ \rangle\!\rangle = To \in \mathscr{F}'$ is a t-operation over $\mathscr{F}$ with a corresponding set of functional elements $\mathscr{B}_0$, then the following analogue to 13.9* shows that t-recursiveness is a particular instance of $\mathscr{B}'$-recursiveness.

Proposition 13.20*. A unary mapping Γ over $\mathscr{F}$ is t-recursive in $\mathscr{B}$ iff Γ is $\check{\mathscr{B}}_0 \cup \{To\}$-recursive in $\mathscr{B}$.

This follows from 12.42*.

Accordingly, the equality

$$\mathscr{R}_0(\mathscr{B}, \check{\mathscr{B}}_0 \cup \{To\}) = \mathscr{R}(\mathscr{B}, \check{\mathscr{B}}_0 \cup \{To\})$$

holds for all $\mathscr{B}$.

There is another possible situation of some interest, viz. consecutive spaces $\mathscr{S}$, $\mathscr{S}'$ with a t-operation $\langle\!\langle\ \ \rangle\!\rangle$ over $\mathscr{F}'$. In particular, such spaces and operations appear in the next chapter (the spaces $\mathscr{S}'$, $\mathscr{S}''$ and the operation Tf), as well as in chapter 28 (the spaces $\mathscr{S}$, $\mathscr{S}_1$ and the operation St). We shall not adduce the relevant general considerations, which are in the spirit of the last two chapters.

A possible direction for further investigations in recursion theory on consecutive spaces is to study $\mathscr{B}'$-recursiveness for certain specific $\mathscr{B}'$. We introduce some monotonic mappings over $\mathscr{F}$ involving quantification, which may be taken to be in $\mathscr{B}'$, provided they are in $\mathscr{F}'$.

Let $\sigma \in \mathscr{F} \setminus \{O\}$ and Q be a *monotonic quantifier* over the set $\mathscr{A}_\sigma = \{\psi / O < \psi \leq \sigma\}$, i.e. $\varnothing \subset Q \subset 2^{\mathscr{A}_\sigma}$ and, if $\mathscr{A} \in Q$ and $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{A}_\sigma$, then $\mathscr{B} \in Q$. Let $Q^\cup$ be the quantifier dual to Q, i.e. $Q^\cup = \{\mathscr{A} / \mathscr{A} \subseteq \mathscr{A}_\sigma \,\&\, \mathscr{A}_\sigma \setminus \mathscr{A} \in Q\}$. Consider the mapping $Q_\sigma : \mathscr{F} \to \mathscr{F}$ such that

$$Q_\sigma(\theta) = \begin{cases} L, & \text{if } Q\psi \in \mathscr{A}_\sigma(\psi \leq \theta), \\ R, & \text{if } \forall \tau \geq \theta Q^\cup \psi \in \mathscr{A}_\sigma(\psi \nleq \tau), \\ O, & \text{otherwise,} \end{cases}$$

where $Q\psi \in \mathscr{A}_\sigma(\text{———})$ stands for $\{\psi / \psi \in \mathscr{A}_\sigma \;\&\; \text{———}\} \in Q$ as usual.

Proposition 13.21. Q_σ is monotonic.

Proof. Suppose that $\theta \leq \theta_1$. If $Q_\sigma(\theta) = O$, then $Q_\sigma(\theta) \leq Q_\sigma(\theta_1)$ trivially. If $Q_\sigma(\theta) = L$, then $Q\psi \in \mathscr{A}_\sigma(\psi \leq \theta)$; hence $Q\psi \in \mathscr{A}_\sigma(\psi \leq \theta_1)$ by the monotonicity of Q; and so $Q_\sigma(\theta_1) = L$ If $Q_\sigma(\theta) = R$, then $\forall \tau \geq \theta Q^\cup \psi \in \mathscr{A}_\sigma(\psi \nleq \tau)$; hence $\forall \tau \geq$

$\theta_1 Q^{\cup}\psi\in\mathscr{A}_\sigma(\psi\not\leq\tau)$, which implies $Q_\sigma(\theta_1)=R$. The proof is complete.

Taking $Q=\exists,\forall$ in particular, one gets

$$E_\sigma(\theta)=\begin{cases} L, & \text{if } \exists\psi(O<\psi\leq\sigma,\theta),\\ R, & \text{if } \forall\tau\geq\theta\forall\psi\neg(O<\psi\leq\sigma,\tau),\\ O, & \text{oherwise}\end{cases}$$

$$A_\sigma(\theta)=\begin{cases} L, & \text{if } \sigma\leq\theta,\\ R, & \text{if } \forall\tau\geq\theta(\sigma\not\leq\tau),\\ O, & \text{otherwise.}\end{cases}$$

The above mappings fit the case when the elements L, R have no upper bound in $\mathscr{F}$. If L, R have an upper bound U, then each two members of $\mathscr{F}$ have an upper bound (see the hint to exercise 7.7); hence the mapping Q_σ becomes

$$Q_\sigma(\theta)=\begin{cases} L, & \text{if } Q\psi\in\mathscr{A}_\sigma(\psi\leq\theta),\\ O, & \text{otherwise.}\end{cases}$$

However, appropriate mappings embodying quantification can be introduced in this instance as well. Take a fixed subset $\mathscr{C}$ of $\mathscr{F}$, $\sigma\in\mathscr{F}\setminus\{O\}$ and a monotonic quantifer Q over $\mathscr{A}_\sigma$. Writing $\mathscr{A}_Q$, $\mathscr{C}_\exists$ respectively for $\{\theta/Q\psi\in\mathscr{A}_\sigma(\psi\leq\theta)\}$ $\{\theta/\exists\psi\in\mathscr{C}(\psi\leq\theta)\}$, the following mapping is monotonic

$$Q_{\sigma,\mathscr{C}}(\theta)=\begin{cases} L, & \text{if } \theta\in\mathscr{A}_Q\,\&\,\forall\tau\leq\theta(\tau\notin\mathscr{C}_\exists\setminus\mathscr{A}_Q),\\ R, & \text{if } \theta\in\mathscr{C}_\exists\setminus\mathscr{A}_Q,\\ U, & \text{if } \theta\in\mathscr{A}_Q\,\&\,\exists\tau\leq\theta(\tau\in\mathscr{C}_\exists\setminus\mathscr{A}_Q),\\ O, & \text{if } \theta\notin\mathscr{C}_\exists\cup\mathscr{A}_Q.\end{cases}$$

Recursion in Q_σ and $Q_{\sigma,\mathscr{C}}$ has been intensively studied in some particular instances for, as will be shown in the exercises below, these mappings generalize the well known functionals $F^{\#}_Q$ and E of Kechris and Moschovakis [1977] and Moschovakis [1969] respectively. In the general theory however, the usefulness of Q_σ and $Q_{\sigma,\mathscr{C}}$ is yet to be tested.

EXERCISES TO CHAPTER 13

Exercise 13.1. Let $\mathscr{S}$ be the IOS of example 4.7 and let $\mathscr{S}'$ be obtained from it by 12.1. Let 0 be a fixed member of M, $\sigma=\lambda s.0$, let Q_0 be a monotonous quantifier over M and let

$$Q=\{\mathscr{A}/\mathscr{A}\subseteq\mathscr{A}_\sigma\,\&\,\cup\{\operatorname{Dom}\psi/\psi\in\mathscr{A}\}\in Q_0\}.$$

Show that $\mathscr{S}$, $\mathscr{S}'$ are consecutive IOS, $Q_\sigma\in\mathscr{F}'$ and

$$Q_\sigma(\theta)=\begin{cases} L, & \text{if } Q_0 s(\theta(s)=0),\\ R, & \text{if } Q_0^{\cup}s(\theta(s)\downarrow\,\&\,\theta(s)\neq 0),\\ O, & \text{otherwise.}\end{cases}$$

Hint. Use exercises 12.1, 12.2, 5.3 to show that $\mathscr{S}'$ is iterative. Employ the equality

$$Q_0=\{X/X\subseteq M\,\&\,\{\psi/O<\psi\leq\sigma\restriction X\}\in Q\}.$$

Exercise 13.1 implies that the mapping Q_σ is nothing else than the functional $F^{\#}_{Q_0}$ of Kechris and Moschovakis [1977],

$$F^{\#}_{Q_0}(f)=\begin{cases}0, & \text{if } Q_0 s(f(s)=0),\\ 1, & \text{if } Q_0^{\smile} s(f(s)\downarrow \,\&\, f(s)\neq 0),\\ \uparrow, & \text{otherwise,}\end{cases}$$

where f ranges over the subset $\{f/f: M \longrightarrow \omega\}$ of $\mathscr{F}$, assuming $\omega \subseteq M$. Indeed, each of Q_σ, $F^{\#}_{Q_0}$ can be expressed by the other. In particular, E_σ corresponds to $F^{\#}_{\exists}=E^{\#}$.

Notice that Q_σ and $Q_\sigma^{\smile}$ can be reduced to each other in the above example.

Exercise 13.2. Let $\mathscr{S}$ be the IOS of example 4.8, let $\mathscr{S}'$ be obtained from it by 12.1, let σ, Q_0, Q be the same as in exercise 13.1, let $U=\lambda s.\{L(s), R(s)\}$ and $\mathscr{C}=\{\varphi/\varphi: M\to M\}$. Prove that $\mathscr{S}$, $\mathscr{S}'$ are consecutive IOS, $Q_{\sigma,\mathscr{C}}\in\mathscr{F}'$ and

$$E_{\sigma,\mathscr{C}}(\theta)=\begin{cases}L, & \text{if } \exists s(0\in\theta(s))\,\&\,\exists s(\theta(s)\uparrow \vee \theta(s)=0),\\ R, & \text{if } \forall s\exists t\neq 0(t\in\theta(s)\,\&\,0\notin\theta(s)),\\ U, & \text{if } \exists s(0\in\theta(s))\,\&\,\forall s\exists t\neq 0\ (t\in\theta(s)),\\ O, & \text{otherwise.}\end{cases}$$

Therefore, $E_{\sigma,\mathscr{C}}$ is in essence the functional $E:\mathscr{F}\longrightarrow\{0,1\}$ of Moschovakis [1969], where

$$\begin{aligned} &0\in E(\theta), && \text{if } \exists s(0\in\theta(s))\,\&\,\forall s(\theta(s)\downarrow),\\ &1\in E(\theta), && \text{if } \forall s\exists t\neq 0\ (t\in\theta(s)).\end{aligned}$$

(To be more precise, $E_{\sigma,\mathscr{C}}$ gives an extension $E^{\#}$ of E such that $0\in E(\theta)$, if $\exists s(0\in\theta(s))$, without requiring $\forall s(\theta(s)\downarrow$.)

If $\mathscr{S}$, $\mathscr{S}'$ are consecutive spaces, then $\mathscr{S}'$ can play the role of $\mathscr{S}$ and monotonic mappings $Q_{\sigma'}$, $Q_{\sigma',\mathscr{C}}$ over $\mathscr{F}'$ can be considered. Take $\sigma'\in\mathscr{F}'\setminus\{O'\}$. We recall that

$$E_{\sigma'}(\theta')=\begin{cases}L' & \text{if } \exists\psi'(O'<\psi'\leq\sigma',\theta'),\\ R', & \text{if } \forall\tau'\geq\theta'\forall\psi'\neg(O'<\psi'\leq\sigma',\tau'),\\ O', & \text{otherwise,}\end{cases}$$

$$A_{\sigma'}(\theta')=\begin{cases}L', & \text{if } \sigma'\leq\theta',\\ R', & \text{if } \forall\tau'\geq\theta'(\sigma'\not\leq\tau'),\\ O', & \text{otherwise.}\end{cases}$$

Exercise13.3. Let $A_\sigma\in\mathscr{F}'$ for all σ. Show that $E_{\sigma'}(\theta')=L'$ iff $\exists\theta(E_{\sigma'(\theta)}(\theta'(\theta))=L)$. In particular, $E_{\dot\sigma}(\theta')=L'$ iff $\exists\theta(E_\sigma(\theta'(\theta))=L)$.

Hint. It suffices to show that $\exists\psi'(O'<\psi'\leq\sigma',\theta')$ iff $\exists\theta\exists\psi(O<\psi\leq\sigma'(\theta),\theta'(\theta))$. Assuming $O<\psi\leq\sigma'(\theta),\theta'(\theta)$, take $\psi'=Ml(A_\theta,\breve\psi,O')$.

Exercise 13.4. Prove that if $E_{\sigma'(\theta)}(\theta'(\theta))=R$ whenever $\sigma'(\theta)\neq O$, then $E_{\sigma'}(\theta')=R'$. In particular, $\forall\theta(E_\sigma(\theta'(\theta))=R)$ implies $E_{\dot\sigma}(\theta')=R'$.

Hint. Show that $\exists\tau' > \theta'\exists\psi'(O' < \psi' < \sigma', \tau')$ implies $\exists\theta\exists\tau > \theta'(\theta)\exists\psi\ (O < \psi \leq \sigma'(\theta), \tau)$.

Exercise 13.5. Show that $A_{\sigma'}(\theta') = L'$ iff $A_{\sigma'(\theta)}\ (\theta'(\theta)) = L$ for all θ such that $\sigma'(\theta) \neq O$. In particular, $A_{\dot\sigma}(\theta') = L'$ iff $\forall\theta(A_\sigma(\theta'(\theta)) = L)$.

Exercise 13.6. Show that $\exists\theta(A_{\sigma'(\theta)}(\theta'(\theta)) = R)$ implies $A_{\sigma'}(\theta') = R'$. In particular, $\exists\theta(A_\sigma(\theta'(\theta)) = R)$ implies $A_{\dot\sigma}(\theta') = R'$.

Hint. Suppose that $\tau' \geq \theta'$ and $\sigma' \leq \tau'$. Then $\sigma'(\theta), \theta'(\theta) \leq \tau'(\theta)$ for all θ; hence $A_{\sigma'(\theta)}(\theta'(\theta)) \neq R$ for all θ such that $\sigma'(\theta) \neq O$, which is not the case.

The last four exercises show that $E_{\sigma'}$, $A_{\sigma'}$ embody $\exists$, $\forall$-quantification over $\mathscr{F}$.

CHAPTER 14

Transfer operation

The present chapter studies a specific t-operation called *transfer*, which is connected with the operator nature of the elements in consecutive spaces.

Suppose that $\mathscr{S}, \mathscr{S}', \mathscr{S}''$ are consecutive IOS (i.e. both $\mathscr{S}, \mathscr{S}'$ and $\mathscr{S}', \mathscr{S}''$ are consecutive) and $Tf: \mathscr{F}'' \to \mathscr{F}''$ satisfies

$$Tf(\varphi'')(\theta')(\theta) = \varphi''(\theta'(L'\check{\theta}, I'))(R\theta)$$

for all $\varphi'' \in \mathscr{F}''$, $\theta' \in \mathscr{F}'$ and $\theta \in \mathscr{F}$. Consecutive spaces to be constructed in chapter 19 admit such an operation.

In order to show that Tf is a storing operation in the sense of chapter 10, take $\mathscr{L}'' = \tilde{\mathscr{F}} = \{\tilde{\rho}/\rho \in \mathscr{F}\}$, where $\tilde{\rho} = \lambda\theta'.\theta'(\check{\rho}, I') = Ml'(I'', \rho^{\vee\vee}, Id') = Ml'(I'', \bar{\rho}Id^{\vee}, Id')$. We recall that Id, $Ml \in \mathscr{F}'$ and Id', $Ml' \in \mathscr{F}''$, where $Id = \lambda\theta.I$, $Ml = \lambda\theta.L\theta R\theta$, $Id' = \lambda\theta'.I'$ and $Ml' = \lambda\theta'.L'\theta' R'\theta'$.

Proposition 14.1. $\tilde{\rho}\overline{\varphi'} = \overline{\varphi'}\tilde{\rho}$.

Proof.

$$\tilde{\rho}\overline{\varphi'}(\theta') = \tilde{\rho}(\overline{\varphi'}(\theta')) = \varphi'\theta'(\check{\rho}, I') = \overline{\varphi'}\tilde{\rho}(\theta')$$

for all θ'; hence $\tilde{\rho}\overline{\varphi'} = \overline{\varphi'}\tilde{\rho}$

Proposition 14.2. $\tilde{\rho}(\varphi'', \psi'') = (\tilde{\rho}\varphi'', \tilde{\rho}\psi'')$.

Proof. Using 14.1 and the equalities $L'' = \overline{L'}$, $R'' = \overline{R'}$, $(L'', R'') = \overline{(L', R')}$, we get

$$\begin{aligned}\tilde{\rho}(\varphi'', \psi'') &= \tilde{\rho}(L'', R'')(\varphi'', \psi'') = (L'', R'')\tilde{\rho}(\varphi'', \psi'') = (L''\tilde{\rho}, R''\tilde{\rho})(\varphi'', \psi'') \\ &= (\tilde{\rho}L'', \tilde{\rho}R'')(\varphi'', \psi'') = (\tilde{\rho}\varphi'', \tilde{\rho}\psi'').\end{aligned}$$

Proposition 14.3. $Tf(\overline{\varphi'}) = \overline{\varphi'}Ml'(I'', \check{L}', \check{R}') = Ml'(\overline{\varphi'}, \check{L}', \check{R}')$. Therefore, $Tf(I'') = Ml'(I'', \check{L}', \check{R}')$ and $Tf(\overline{\varphi'}) = \overline{\varphi'}\,Tf(I'') = Tf(I'')\overline{\varphi'}$.

Proof. We have

$$Tf(\overline{\varphi'})(\theta')(\theta) = \varphi'\theta'(L'\check{\theta}, I')(R\theta) = \varphi'\theta'(L', R')(\theta)$$

for all θ', θ; hence $Tf(\overline{\varphi'})(\theta') = \varphi'\theta'(L', R')$ for all θ', which completes the proof.

Proposition 14.4. $Tf(\breve{\varphi}') = \overline{\varphi'}\breve{R}'$.

Proof.

$$Tf(\breve{\varphi}')(\theta')(\theta) = \varphi'(R\theta) = \varphi' R'(\theta)$$

for all θ', θ, hence $Tf(\breve{\varphi}')(\theta') = \varphi' R'$ for all θ'.

Proposition 14.5. $\tilde{\rho} Tf(\varphi'') = \varphi''\tilde{\rho}$.

Proof.

$$\tilde{\rho} Tf(\varphi'')(\theta')(\theta) = \tilde{\rho}(Tf(\varphi'')(\theta'))(\theta) = Tf(\varphi'')(\theta')((\rho,\theta)) = \varphi''(\theta'(\breve{\rho}, I'))(\theta)$$

for all θ', θ; hence $\tilde{\rho} Tf(\varphi'')(\theta') = \varphi''(\theta'(\breve{\rho}, I')) = \varphi''\tilde{\rho}(\theta')$ for all θ'. The proof is complete.

Proposition 14.6. Whenever $\varphi''\tilde{\rho}\psi'' \leq \chi''\tilde{\rho}\tau''$ for all ρ, then $Tf(\varphi'')\psi'' \leq Tf(\chi'')\tau''$.

Proof. We have

$$\begin{aligned} T(\varphi'')\psi''(\theta')(\theta) &= \varphi''(\psi''(\theta')(L'\breve{\theta}, I'))(R\theta) = \varphi''(L\theta)^{\sim}\psi''(\theta')(R\theta) \\ &\leq \chi''(L\theta)^{\sim}\tau''(\theta')(R\theta) = Tf(\chi'')\tau''(\theta')(\theta) \end{aligned}$$

for all θ', θ, which completes the proof.

Now take

$$K_0'' = Tf(I''),\ K_1'' = MI'(I'', (\breve{L}', L''\breve{R}'), R''\breve{R}'),\ K_2'' = MI'(I'', L''\breve{L}', R''\breve{L}', \breve{R}').$$

Proposition 14.7. $K_0''(\varphi'', \psi'') = (K_0''\varphi'', K_0''\psi'')$.

This follows from the proof of 14.2 since $K_0''\overline{\varphi'} = \overline{\varphi'}K_0''$ by 14.3.

Proposition 14.8. $\tilde{\rho}K_0''(L'', R'') = (\tilde{\rho}L'', \tilde{\rho}R'')$.

Proof. 14.5, 14.2 give

$$\tilde{\rho}K_0''(L'', R'') = \tilde{\rho}(L'', R'') = (\tilde{\rho}L'', \tilde{\rho}R'').$$

Proposition 14.9. $\tilde{\rho}\tilde{\sigma}K_1'' = (\sigma,\rho)^{\sim}$, $(\sigma,\rho)^{\sim}K_2'' = \tilde{\rho}\tilde{\sigma}$.

Proof. We have

$$\begin{aligned} \tilde{\rho}\tilde{\sigma}K_1''(\theta') &= \tilde{\sigma}K_1''(\theta')(\breve{\rho}, I') = K_1''(\theta')(\breve{\sigma}, I')(\breve{\rho}, I') = \theta'((L', L'R'), R'^2)(\breve{\sigma}, \breve{\rho}, I') \\ &= \theta'((\breve{\sigma}, \breve{\rho}), I') = \theta'((\sigma,\rho)^{\breve{}}, I') = (\sigma,\rho)^{\sim}(\theta') \end{aligned}$$

for all θ'. On the other hand,

$$\begin{aligned} (\sigma,\rho)^{\sim}K_2''(\theta') &= K_2''(\theta')((\sigma,\rho)^{\breve{}}, I') = \theta'(L'^2, R'L', R')((\breve{\sigma}, \breve{\rho}), I') \\ &= \theta'(\breve{\sigma}, \breve{\rho}, I') = \tilde{\rho}\tilde{\sigma}(\theta') \end{aligned}$$

for all θ'. The proof is complete.

Notice that $Tf(I'')K_0'' = K_0''$ by 14.6, while $Tf(Tf(I''))K_1'' = K_1''$ is valid as well.

Proposition 14.10. Tf is a t-operation with functional elements $K_0'' - K_2''$, $K_3'' = \tilde{I}$, $K_4'' = MI'(I'', \breve{R}')$.

Proof. Transfer is obviously monotonic and 14.8, 14.9, 14.5, 14.6 imply that it is a storing operation. The equality $\tilde{\rho}K_4''(\theta') = \theta' R'(\check{\rho}, I') = \theta'$ gives $\tilde{\rho}K_4'' = I''$, which completes the proof by 10.18.

Proposition 14.11. $\overline{\varphi'}\, Tf(\psi'') = Tf(\overline{\varphi'}\psi''), Tf(\psi'')\overline{\varphi'} = Tf(\psi''\overline{\varphi'})$.
This follows from 14.3, 10.12.

Proposition 14.12. $Tf((\varphi'', \psi'')) = (Tf(\varphi''), Tf(\psi''))$, $Tf(\Delta(\varphi'', \psi'')) = \Delta(Tf(\varphi''), Tf(\psi''))$, $Tf([\varphi'']) = [Tf(\varphi'')]Tf(I'')$. In particular, $Tf(\langle \varphi'' \rangle) = \Delta(Tf(\varphi'' L''), Tf(R''))$.

Proof. The first equality follows from 10.13, 14.7, 10.12. Using 6.34, one gets $\Delta(Tf(\varphi''), Tf(\psi'')) = \langle I'' \rangle Tf(\Delta(\varphi'', \psi''))$ which implies the second equality by 14.11. The third equality follows by the first one and the proof of 10.15.

Proposition 14.13. $\varphi'' = Ml'(I'', Id', Id')Tf(\varphi'')K_4''$, hence K_3'' can be replaced by $Ml'(I'', Id', Id')$.

Proof. We have

$$\begin{aligned} Ml'(I'', Id', Id')Tf(\varphi'')K_4''(\theta')(\theta) &= Tf(\varphi'')(\theta' R')(I', I')(\theta) \\ &= Tf(\varphi'')(\theta' R')((\theta, \theta)) = \varphi''(\theta')(\theta) \end{aligned}$$

for all θ', θ. The proof is complete.

Proposition 14.14. $Tf(Id') = R'' Id'$.
This follows from 14.4.

Proposition 14.15. $Tf(Ml') = Ml'(L'', \check{L}', Ml'(R'', \check{L}', \check{R}'))$.

Proof.

$$\begin{aligned} Tf(Ml')(\theta')(\theta) &= Ml'(\theta'(L'\check{\theta}, I'))(R\theta) = L'\theta'(L'\check{\theta}, I')R'\theta'(L'\check{\theta}, I')(R\theta) \\ &= L'\theta'(L'\check{\theta}, R'\theta'(L'\check{\theta}, I'))(R\theta) = L'\theta'(L', R'\theta'(L', R'))(\theta) \\ &= Ml'(L'', \check{L}', Ml'(R'', \check{L}', \check{R}'))(\theta')(\theta) \end{aligned}$$

for all θ', θ, which completes the proof.

Proposition 14.10 can be sharpened as follows.

Proposition 14.16 (Transfer Operation Theorem). Tf is a t-operation with a set of functional elements $\mathscr{B}_0'' = \{L'', R'', Id', Ml'\}$ and mappings Σ_0'', Σ_1'', $\Sigma_3'' - \Sigma_5''$ prime recursive in $\mathscr{B}_0''$.

Proof. The elements I'', $Tf(I'')$, $Tf(L'')$, $Tf(R'')$, $Tf(Id')$, $Tf(Ml')$ may be removed from $\mathscr{B}_0''$ since they are prime recursive in Id', Ml' by 5.12, 14.3, 14.14, 14.15. The mappings $\Sigma_0'', \Sigma_1'', \Sigma_3'' - \Sigma_5''$ satisfying the equalities (0), (1), (3)–(5) of chapter 10 by 10.12, 14.12, 10.16, 14.13 are prime recursive in $\mathscr{B}_0''$. This completes the proof.

To indicate that notions of chapter 10 concern the operation Tf now, we write tf-recursive, tf-recursive, tf-simple segment, tf μ-axiom etc. Recall that φ'' is tf-recursive in $\mathscr{B}'' \subseteq \mathscr{F}''$ iff

$$\varphi'' \in cl(\mathscr{B}_0'' \cup \mathscr{B}''/\circ, \Pi, \langle\quad\rangle, [\quad], Tf).$$

Proposition 14.16 allows to apply the theory developed in chapter 10 to transfer operation and tf-recursiveness. While the proof of the First tf-Recursion Theorem depends on tfμA, the verification of this axiom in particular instances is simplified by the fact that all tf-simple segments are normal by 10.18.

The spaces $\mathscr{S}'$, $\mathscr{S}''$ are consecutive, hence tf-recursiveness can be characterized by making use of 10.2, 14.4, 12.34 as follows.

Proposition 14.17. Let $\mathscr{B}' \subseteq \mathscr{F}'$, $\mathscr{B}'' \subseteq \mathscr{F}''$. Then the following are equivalent.

(1) φ'' is tf-recursive in $\check{\mathscr{B}}' \cup \mathscr{B}''$.

(2) φ'' is recursive in $\{Id', MI'\} \cup \check{\mathscr{B}}' \cup Tf(\mathscr{B}'')$.

(3) φ'' is prime recursive in $\{Id', MI'\} \cup \check{\mathscr{B}}' \cup \langle Tf(\mathscr{B}'') \rangle$.

In particular, φ'' is tf-recursive in $\check{\mathscr{B}}'$ iff it is prime recursive in $\{Id', MI'\} \cup \check{\mathscr{B}}'$.

Now assume that another IOS $\mathscr{S}'''$ is also given, such that $\mathscr{S}''$, $\mathscr{S}'''$ are consecutive and $Tf \in \mathscr{F}'''$. Then propositions 12.40–12.42* hold with $\mathscr{S}''$, $\mathscr{S}'''$, Tf playing the role of $\mathscr{S}$, $\mathscr{S}'$, $\langle\ \rangle$. Assume finally that $Tf' : \mathscr{F}''' \to \mathscr{F}'''$ is a transfer operation, i.e.

$$Tf'(\varphi''')(\theta'')(\theta') = \varphi'''(\theta''(L''\check{\theta}', I''))(R'\theta')$$

for all $\varphi''' \in \mathscr{F}'''$, $\theta'' \in \mathscr{F}''$, $\theta' \in \mathscr{F}'$. It is of interest to see how do the operations Tf, Tf' interact. Notice that unlike the basic IOS-operations, the isomorphism of $\mathscr{S}''$, $\overline{\mathscr{S}'''}$ does not extent to ensure $\overline{Tf(\varphi'')} = Tf'(\overline{\varphi''})$ for all φ'': take $\varphi'' = Id'$ for example.

Proposition 14.18. $Tf \notin \overline{\mathscr{F}''}$.

Proof. Suppose that $Tf = \overline{\varphi''}$. Then $\varphi'' = \overline{\varphi''}(I'') = Tf(I'')$. Using 14.14, 14.3, we get

$$\check{R}' = Tf(Id') = \overline{\varphi''}\,(Id') = \varphi'' Id' = Tf(I'')Id' = (\check{L}', \check{R}'),$$

which is not the case.

Proposition 14.19. $Tf'(Tf) = MI''(Tf, \overline{MI'}(\check{L}'', R'''I\check{d}'), \check{R}'')$.

Proof. We have

$$\begin{aligned} Tf'(Tf)(\varphi'')(\theta')(\theta) &= Tf(\varphi''(L''\check{\theta}', I''))(R'\theta')(\theta) = \varphi''(L''\check{\theta}', I'')(R'\theta'(L'\check{\theta}, I'))(R\theta) \\ &= \varphi''((L'\theta', R'\theta'(L'\check{\theta}, I')))(R\theta) \\ &= \varphi''((L'\theta'R', R'\theta')(L'\check{\theta}, I'))(R\theta) \\ &= Tf(\varphi'')((L'\theta'R', R'\theta'))(\theta) \end{aligned}$$

for all φ'', θ', θ; hence

$$Tf'(Tf)(\varphi'')(\theta') = Tf(\varphi'')((L'\theta'R', R'\theta')) = Tf(\varphi'')(MI'(L'', \check{R}'), R'')(\theta')$$

for all φ'', θ'. Therefore

$$Tf'(Tf)(\varphi'') = Tf(\varphi'')(MI'(L'', \check{R}'), R'') = MI''(Tf, \overline{MI'}(\check{L}'', R'''I\check{d}'), \check{R}'')(\varphi'')$$

for all φ'', which completes the proof.

Proposition 12.42* can be sharpened to give the following tf-Conservativeness Theorem.

Proposition 14.20*. Let $\mathscr{B}'' \subseteq \mathscr{F}''$ and Γ'' be a unary mapping over $\mathscr{F}''$. Then the following are equivalent.

(1) Γ'' is a mapping tf-recursive in $\mathscr{B}''$.

(2) Γ'' is a member of $\mathscr{F}'''$ prime recursive in

$$\{\check{Id}', \check{Ml}', Id'', Ml'', Tf\} \cup \check{\mathscr{B}}''.$$

(3) Γ'' is a member of $\mathscr{F}'''$ tf'-recursive in

$$\{\check{Id}', \check{Ml}', Tf\} \cup \check{\mathscr{B}}''.$$

Proof. Propositions 14.19, 12.40 imply that the element $\langle Tf'(Tf)\rangle$ is prime recursive in $\check{Id}'$, $\check{Ml}'$, Id'', Ml'', Tf, hence (3) implies (2) by 14.17. The implication (2)$\Rightarrow$(1) follows by 12.42*, while (1)$\Rightarrow$(3) is immediate. This completes the proof.

Using 12.3, 12.19, $\overline{Id'}$, $\overline{Ml'}$, $\overline{\mathscr{B}''}$ can be substituted respectively for $\check{Id}'$, $\check{Ml}'$, $\check{\mathscr{B}}''$ in 14.20* since Id'', Ml'' are among the initial elements.

EXERCISES TO CHAPTER 14

Exercise 14.1. Let $\Pi^*: \mathscr{F}''^2 \to \mathscr{F}''$ be such that

$$\Pi^*(\varphi'', \psi'') = \lambda\theta'.\lambda\theta.\varphi''(\lambda\rho.\psi''(\lambda\sigma.\theta'((\rho,\sigma)))(\theta))(\theta).$$

Show that Tf can be expressed in terms of Π^*, Ml', Id'.

Hint.

$$Tf(\varphi'') = \Pi^*(Ml'(I'', \check{L}'), Ml'(\varphi'', \check{R}')).$$

Exercise 14.2. Show that the operation Π^* of the previous exercise can be expressed in terms of Tf, Ml', Id'.

Hint.

$$\Pi^*(\varphi'', \psi'') = Ml'(Tf(\varphi'')Ml'(Tf(\psi''), \check{R}', \check{L}'), Id', Id').$$

Exercise 14.3.** Show that the t-operation Tf satisfies the assumptions of exercises 10.2–10.5, provided the elements $\check{Id}$, $\check{Ml}$ are added to $\mathscr{B}''_0$.

Hint. Take $\rho = \Delta((L, L^2R, R^2),\ (L, RLR, R^2))$, $\psi''_0 = \langle Ml'\rangle\bar{\rho}(I'', \check{Id}, Id')$ and $\psi''_1 = Tf(Ml')(\check{Ml}, Ml'(I'', \check{R}'))$. These elements are prime recursive in $\check{Id}$, $\check{Ml}$, Id', Ml' by 12.31, 12.33, 14.15. Show that $\bar{n}\psi''_0 = \tilde{\bar{n}}$, $\tilde{\bar{n}}\psi''_1 = \bar{n}$, $\bar{n}$ standing for LR^n or L^nR'''. Use exercise 10.8**.

Exercise 14.4.** Let $\mathscr{B}' \subseteq \mathscr{F}'$, $\mathscr{B}'' \subseteq \mathscr{F}''$. Show that the following are equivalent.

(1) φ'' is tf-recursive in $\{\check{Id}, \check{Ml}\} \cup \check{\mathscr{B}}' \cup \mathscr{B}''$.

(2) φ'' is prime recursive in $\{\check{Id}, \check{Ml}, Id', Ml'\} \cup \check{\mathscr{B}}' \cup Tf(\mathscr{B}'')$.

Hint. Use 14.17, exercises 14.3**, 10.2 and proposition 10.16.

Exercise 14.5. Determine elements K''_4, K''_5, K''_6 to satisfy the assumptions of exercise 10.9 with $\tilde{\bar{\mathscr{F}}}$ playing the role of $\mathscr{L}$.

Hint. Take $K''_4 = Ml'(I'', \check{R}')$, $K''_5 = Ml'(I'', \check{L}', \check{L}, \check{R}')$, $K''_6 = Ml'(I'', L''\check{R}', \check{L}, R''\check{R}')$.

Exercise 14.6. Let $Co: \mathscr{F}^{\omega} \to \mathscr{F}$, $Co': \mathscr{F}'^{\omega} \to \mathscr{F}'$ and $Co'': \mathscr{F}''^{\omega} \to \mathscr{F}''$ be collection operations such that $Co''\{\varphi''_n\}(\theta') = Co'\{\varphi''_n(\theta')\}$ for all $\theta' \in \mathscr{F}'$ and all sequences $\{\varphi''_n\}$ in $\mathscr{F}''$. Determine an element K''_7 to satisfy the assumptions of exercise 11.7 with $\mathscr{L}''_0 = \mathscr{L}'' = \tilde{\bar{\mathscr{F}}}$.

Hint. Take the element ψ''_1 of exercise 14.3*, then take $K''_7 = [I'']Co''\{Ml'(K''_2\psi''_1, \bar{n}\check{L}', \check{R}')\}$ and assign $Co\{(\overline{k_n}, \rho_n)\}^{\sim}\bar{0}$ to $\{\tilde{\rho}_n\overline{k_n}\}$.

Exercise 14.7.** Let $\mathscr{B}'' \subseteq \mathscr{F}''$, $\mathscr{B}''' \subseteq \mathscr{F}'''$ and $\varphi''' \in \mathscr{F}'''$. Show that the following are equivalent.

(1) φ''' is tf-recursive in $\{I\check{d}', M\check{l}', Tf\} \cup \check{\mathscr{B}}'' \cup \mathscr{B}'''$.

(2) φ''' is prime recursive in

$$\{I\check{d}', M\check{l}', Id'', Ml'', Tf\} \cup \check{\mathscr{B}}'' \cup Tf'(\mathscr{B}''').$$

Hint. Use exercise 14.4** and 14.19 to get (1)$\Rightarrow$(2).

Exercise 14.8. Let Tf^* be an operation over $\mathscr{F}'''$ such that

$$Tf^*(\varphi''') = \lambda\theta''.\lambda\theta'.\lambda\theta.\varphi'''(\lambda\tau'.\theta''(\tau')(L'\check{\theta}, I'))(\theta')(R\theta).$$

Show that Tf^* is a storing operation and $Tf^*(\overline{\varphi''}) = \overline{Tf(\varphi'')}$ for all φ''.

Hint. Take $\mathscr{L}^* = \bar{\bar{\tilde{\mathscr{F}}}}$ and $K^*_i = \bar{K}''_i$, $i = 0, 1, 2$.

Exercise 14.9. Show that the operation Tf^* of the previous exercise can be expressed in terms of Tf', Tf.

Hint. Take $\sigma''' = \overline{Ml'}(Ml''(I''', \check{R}''), \check{L}'')$ and show that $Tf^*(\varphi''') = Ml''(Tf\,Tf'(\varphi''')\sigma''', I\check{d}', \overline{Ml'}(Id'', R'^{\sim}))$.

Applying the construction of 12.1 to example 3.1 thrice, one gets consecutive IOS $\mathscr{S}$, $\mathscr{S}'$, $\mathscr{S}''$, $\mathscr{S}'''$ which admit transfer operations Tf, Tf' and collection operations Co, Co', Co'' (by exercise 11.1) satisfying all the assumptions made in this chapter.

CHAPTER 15

Hierarchies of operative spaces

The study of consecutive spaces naturally develops into the study of hierarchies of spaces. For, given an IOS $\mathscr{S}_0$, one is tempted to construct by 12.1 a sequence $\{\mathscr{S}_n\}$ of consecutive spaces, then to try to bring them all together into a single space $\mathscr{S}_\omega$ and iterate, thereby constructing a transfinite sequence of spaces. We feel that such hierarchies of spaces provide the appropriate domains for an intrinsic generalized recursion theory. In this chapter we aim only to establish some basic properties.

Let $\mathscr{S}$ be an IOS. A *hierarchy of* IOS *based on* $\mathscr{S}$ is a transfinite sequence $\{\mathscr{S}_\xi\}$ of IOS such that $\mathscr{S}_0 = \mathscr{S}$ and whenever $\xi < \eta$, then $\mathscr{S}_\xi$ is (isomorphic to) a proper subspace of $\mathscr{S}_\eta$. Hierarchies $\{\mathscr{S}_\xi\}_{\xi<\xi_0}$ of length ξ_0 could also be of interest, allowing us to work either in the single space $\mathscr{S}_{\xi_0}$, if ξ_0 is a limit, or in $\mathscr{S}_\xi$, if $\xi_0 = \xi_1 + 1$.

A hierarchy is called *monotonic* if for all ξ the IOS $\mathscr{S}_\xi, \mathscr{S}_{\xi+1}$ are consecutive and $\mathscr{F}_{\xi_1} = \bigcup_{\eta<\xi} \mathscr{F}_\eta$ whenever $\xi > 0$ is limit, identifying (this is crucial!) the subspace $\bar{\mathscr{F}}_\xi$ of $\mathscr{S}_{\xi+1}$ with $\mathscr{S}_\xi$ for all ξ. Two constructions yielding monotonic hierarchies will be given in chapter 20. They really exploit the idea outlined above, except that beyond ω one takes in $\mathscr{F}_{\xi+1}$ only *some* of the monotonic unary mappings over $\mathscr{F}_\xi$ as opposed to *all* of them.

Now assume that a monotonic hierarchy $\{\mathscr{S}_\xi\}$ is given. The proper class $\bigcup_\xi \mathscr{F}_\xi$ is denoted by $\mathscr{M}\mathscr{H}$ (for 'monotonic hierarchy'). The operations of multiplication, pairing, translation and iteration are well defined on the whole class $\mathscr{M}\mathscr{H}$, so $\mathscr{M}\mathscr{H}$ could be regarded as the carrier of a single 'universal' IOS. Notice that if $\mathscr{B} \subseteq \mathscr{M}\mathscr{H}$ is a set, then cardinality considerations imply that $\mathscr{B} \subseteq \mathscr{F}_\xi$ for a certain ξ. For all $\varphi \in \mathscr{M}\mathscr{H}$ the ordinal $r(\varphi) = \min\{\xi/\varphi \in \mathscr{F}_\xi\}$ (which is zero or a successor) is called the *rank of* φ. Accordingly, $\varphi \in \mathscr{F}_\xi$ iff $r(\varphi) \leq \xi$.

If $\varphi \in \mathscr{F}_{\xi+1}$, $\psi \in \mathscr{F}_\xi$ and φ_1 is the member of $\mathscr{F}_{\xi+1}$ identified with φ, we write $\varphi(\psi)_\xi$ for $\varphi_1(\psi)$. It follows that $\varphi(\psi)_\xi = \varphi(\psi)$, provided $r(\varphi) = \xi + 1$. If $r(\varphi) < \xi + 1$, then a certain member φ_2 of $\mathscr{F}_\xi$ is identified with φ so that $\varphi_1 = \overline{\varphi_2}$; hence $\varphi(\psi)_\xi = \overline{\varphi_2}(\psi) = \varphi_2\psi$, which gives $\varphi(\psi)_\xi = \varphi\psi$. In particular, if $\varphi \in \mathscr{F}_\xi$, then $\varphi = \varphi(I)_\xi$; this will often be used in the sequel. Moreover, $\varphi, \psi \in \mathscr{F}_{\xi+1}$, $\chi \in \mathscr{F}_\xi$ and $\varphi \leq \psi$ imply $\varphi(\chi)_\xi \leq \psi(\chi)_\xi$.

Another convenient notation is $\lambda_\xi\theta.$——— for the mapping which assigns to each $\theta \in \mathscr{F}_\xi$ the expression———.

Let Id_ξ be the mapping Id of chapter 12 corresponding to $\mathscr{S}_\xi$. Then

$Id_\xi = \lambda_\xi\theta.I$ and $Id_\xi \in \mathscr{F}_{\xi+1}$. On the other hand, 12.5 implies that $Id_\xi \notin \mathscr{F}_\xi$; hence $r(Id_\xi) = \xi + 1$.

Proposition 15.1. $Id_\xi\varphi = Id_\xi$ for all $\varphi \in \mathscr{M}\mathscr{H}$.

Proof. By transfinite induction on $r(\varphi)$.

If $r(\varphi) \leq \xi + 1$, then $\varphi \in \mathscr{F}_{\xi+1}$ and $Id_\xi\varphi = Id_\xi$ follows from 12.4.

Suppose that $r(\varphi) = \eta + 1 > \xi + 1$ and $Id_\xi\psi = Id_\xi$ whenever $r(\psi) < \eta + 1$. Let $\theta \in \mathscr{F}_\eta$. Then $Id_\xi\varphi \in \mathscr{F}_{\eta+1}$, $\varphi(\theta)_\eta \in \mathscr{F}_\eta$, hence $Id_\xi\varphi(\theta)_\eta = Id_\xi(\varphi(\theta)_\eta)_\eta = Id_\xi \circ (\varphi(\theta)_\eta)$ since $r(Id_\xi) < \eta + 1$. The induction assumption gives $Id_\xi \circ (\varphi(\theta)_\eta) = Id_\xi$; hence

$$Id_\xi\varphi(\theta)_\eta = Id_\xi = Id_\xi\theta = Id_\xi(\theta)_\eta.$$

This holds for all $\theta \in \mathscr{F}_\eta$; hence $Id_\xi\varphi = Id_\xi$. This completes the proof.

Proposition 15.2. If $\varphi \in \mathscr{F}_\xi$, then $\varphi Id_\xi = \lambda_\xi\theta.\varphi$. If $\varphi \in \mathscr{F}_{\xi+1}$, then $\varphi Id_\xi = \varphi(I)_\xi Id_\xi$. If $\varphi \in \mathscr{F}_{\eta+1}$, $\eta > \xi$, then $\varphi Id_\xi = \varphi(Id_\xi)_\eta Id_\eta$.

Proof. The first two assertions follows from 12.3 and 12.14.

Whenever $\varphi \in \mathscr{F}_{\eta+1}$, $\eta > \xi$, then $\varphi Id_\xi \in \mathscr{F}_{\eta+1}$, hence

$$\begin{aligned}\varphi Id_\xi &= \lambda_\eta\theta.(\varphi Id_\xi)(\theta)_\eta = \lambda_\eta\theta.\varphi(Id_\xi(\theta)_\eta)_\eta = \lambda_\eta\theta.\varphi(Id_\xi\theta)_\eta \\ &= \lambda_\eta\theta.\varphi(Id_\xi)_\eta = \varphi(Id_\xi)_\eta Id_\eta\end{aligned}$$

using 15.1. The proof is complete.

Let Ml_ξ be the mapping Ml of chapter 12 corresponding to $\mathscr{S}_\xi$. Then $Ml_\xi = \lambda_\xi\theta.L\theta R\theta$ and $r(Ml_\xi) = \xi + 1$ by 12.17.

Proposition 15.3. If φ, $\psi \in \mathscr{F}_{\xi+1}$, then $Ml_\xi(\varphi, \psi) = \lambda_\xi\theta.\varphi(\theta)_\xi\psi(\theta)_\xi$. In particular, if φ, $\psi \in \mathscr{F}_\xi$, then $Ml_\xi(\varphi Id_\xi, \psi Id_\xi) = \varphi\psi Id_\xi$.

This follows from 12.16, 12.20.

Proposition 15.4. If $\varphi \in \mathscr{F}_{\xi+1}$, $\psi \in \mathscr{F}_\xi$, then $\varphi(\psi)_\xi = Ml_\xi(\varphi\psi Id_\xi, I)$.

Proof. Using 12.21, we get

$$Ml_\xi(\varphi\psi Id_\xi, I) = (\varphi\psi)(I)_\xi = \varphi(\psi(I)_\xi)_\xi = \varphi(\psi)_\xi.$$

Let Tr_ξ, It_ξ be the mappings Tr, It of chapter 12 corresponding to $\mathscr{S}_\xi$. Then $Tr_\xi = \lambda_\xi\theta.\langle\theta\rangle$, $It_\xi = \lambda_\xi\theta.[\theta]$ and $r(Tr_\xi) = r(It_\xi) = \xi + 1$ by 12.30.

We are now going to specify abstract concepts of effective computability within the hierarchy, bearing in mind those introduced in chapter 7.

Let $\varphi \in \mathscr{M}\mathscr{H}$ and $\mathscr{B} \subseteq \mathscr{M}\mathscr{H}$. Then the notions of relative recursiveness, prime recursiveness, primitive recursiveness, primitiveness and polynomiality reappear as follows. We say that φ is *recursive* (*prime recursive* etc.) *in* $\mathscr{B}$ iff $\varphi \in \mathscr{F}_\xi$ and φ is recursive (prime recursive etc.) in $\mathscr{B} \cap \mathscr{F}_\xi$ in the sense of $\mathscr{S}_\xi$ for a certain ξ. This is a sound definition since if $\xi \leq \eta$, then $\mathscr{S}_\xi$ is a subspace of $\mathscr{S}_\eta$ and $\mathscr{B} \cap \mathscr{F}_\xi \subseteq \mathscr{F}_\xi$; hence $\varphi \in \mathscr{F}_\xi$ is recursive (prime recursive etc.) in $\mathscr{B} \cap \mathscr{F}_\xi$ in the sense of $\mathscr{S}_\xi$ iff φ is recursive (prime recursive etc.) in $\mathscr{B} \cap \mathscr{F}_\xi$ in the sense of $\mathscr{S}_\eta$.

The above notions satisfy the ordinary properties of chapter 7 since those properties hold for $\mathscr{S}_\xi$ for all ξ. In particular it is worth mentioning that whenever φ is recursive (prime recursive etc.) in $\mathscr{B}$, then φ is recursive (prime recursive etc.) in a finite subclass of $\mathscr{B}$.

Proposition 15.5. The elements Tr_ξ, It_ξ are prime recursive in Id_ξ, Ml_ξ.

This follows from 12.28, 12.29.

Proposition 15.6. Let $\varphi \in \mathscr{F}_\xi$ and $\mathscr{B} \subseteq \mathscr{F}_\xi$. Then the following are equivalent.

(1) φ is recursive in $\mathscr{B}$.

(2) $\varphi \in cl(\mathscr{B} \cup \{L, R, Id_\xi, Ml_\xi, Tr_\xi, It_\xi\}/\circ, \Pi)$.

(3) $\varphi \in cl(\mathscr{B} \cup \{L, A, Id_\xi, Ml_\xi, Tr_\xi, It_\xi\}/\circ)$.

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are immediate. Supposing (3), we get that φ is a mapping over $\mathscr{F}_\xi$ recursive in $\mathscr{B}$, hence $\varphi = \varphi(I)_\xi$ is a member of $\mathscr{F}_\xi$ recursive in $\mathscr{B}$. The proof is complete.

A version of 15.6 for prime recursiveness can be given with Tr_ξ omitted, and similarly for the other notions considered. Further on we shall concentrate on recursiveness, taking the element A as initial and correspondingly omitting R and the pairing operation.

The following Hierarchy Conservativeness Theorem is an extension of 12.38*.

Proposition 15.7*. Let $\varphi \in \mathscr{F}_\xi$, $\mathscr{B} \subseteq \mathscr{F}_\xi$ and φ be recursive in $\mathscr{B} \cup \{Id_\eta, Ml_\eta/\eta \geq \xi\}$. Then φ is recursive in $\mathscr{B}$.

Proof. Writing $\mathscr{B}_\eta$ for $\mathscr{B} \cup \{Id_\zeta, Ml_\zeta/\xi \leq \zeta \leq \eta\}$, there is a η_0 such that φ is recursive in $\mathscr{B}_{\eta_0}$. Suppose that $\eta > \xi$. Applying 12.38* to the consecutive spaces $\mathscr{S}_{\eta_0}, \mathscr{S}_{\eta_0+1}$, we see that φ is a mapping over $\mathscr{F}_{\eta_0}$ recursive in $\mathscr{B} \cup \{Id_\zeta, Ml_\zeta/\xi \leq \zeta < \eta_0\}$; hence $\varphi = \varphi(I)_{\eta_0}$ is an element recursive in $\mathscr{B} \cup \{Id_\zeta, Ml_\zeta/\xi \leq \zeta < \eta_0\}$. Therefore, φ is recursive in $\mathscr{B}_{\eta_1}$ for a certain $\eta_1 < \eta_0$. Since any decreasing sequence of ordinals is finite, we obtain in finitely many steps an $\eta_n \leq \xi$ such that φ is recursive in $\mathscr{B}_{\eta_n}$. Therefore φ is recursive in $\mathscr{B} \cup \{Id_\xi, Ml_\xi\}$; hence $\varphi = \varphi(I)_\xi$ is recursive in $\mathscr{B}$ by 12.38*. The proof is complete.

By 15.5, $\{Id_\eta, Ml_\eta/\eta \geq \xi\}$ can be replaced by $\{Id_\eta, Ml_\eta, Tr_\eta, It_\eta/\eta \geq \xi\}$.

The following conservativeness result called the Hierarchy First Recursion Theorem is in a sense analogous to the First Recursion Theorem 5.3.2 of Platek [1966].

Proposition 15.8*. Let $\varphi \in \mathscr{F}_{\xi+1}$ and $\mathscr{B} \subseteq \mathscr{F}_\xi$. Then the following are equivalent.

(1) φ is (prime) recursive in $\mathscr{B} \cup \{Id_\xi, Ml_\xi\}$.

(2) $\varphi \in cl(\mathscr{B} \cup \{L, A, Id_\xi, Ml_\xi, Tr_\xi, It_\xi\}/\circ)$.

(3) $\varphi \in cl(\mathscr{B} \cup \{L, A, Id_\eta, Ml_\eta, Tr_\eta, It_\eta/\eta \geq \xi\}/\circ)$.

If $\varphi \in \mathscr{F}_\xi$, then (1) can be replaced by 'φ is recursive in $\mathscr{B}$'.

Proof. The sentence (1) means that φ is (prime) recursive in $\mathscr{B} \cup \{Id_\xi, Ml_\xi\}$ within $\mathscr{S}_{\xi+1}$. Therefore, (1) and (2) are equivalent by the Conservativeness Theorem 12.38*. It is immediate that (2) implies (3), while (3) implies (2) by 15.7*.

If $\varphi \in \mathscr{F}_\xi$, then (1) implies that φ is recursive in $\mathscr{B}$ by 12.38*. The proof is complete.

Now assume that the hierarchy $\{\mathscr{S}_\xi\}$ admits the operation Tf of chapter 14, i.e. for all ξ the mapping.

$$Tf_{\xi+2} = \lambda_{\xi+2}\varphi . \lambda_{\xi+1}\rho . \lambda_\xi\theta . \varphi(\rho(L\theta Id_\xi, I))_{\xi+1}(R\theta)_\xi$$

is a member of $\mathscr{F}_{\xi+3}$. Of course, $r(Tf_{\xi+2}) = \xi + 3$ by 14.18. The hierarchies to be constructed in chapter 20 admit transfers.

Let $\varphi \in \mathscr{M}\mathscr{H}$ and $\mathscr{B} \subseteq \mathscr{M}\mathscr{H}$. Then we say that φ is *$tf_{\xi+2}$-recursive in $\mathscr{B}$* iff $\varphi \in \mathscr{F}_{\xi+2}$ and φ is tf-recursive in $\mathscr{B} \cap \mathscr{F}_{\xi+2}$ in the sense of chapter 14. If φ is $tf_{\xi+2}$-recursive in $\mathscr{B}$ and $\xi < \eta$, then it can not be claimed that φ is also $tf_{\eta+2}$-recursive in $\mathscr{B}$. Instead, φ is (prime) recursive in $\mathscr{B} \cup \{Id_{\xi+1}, Ml_{\xi+1}, Tf_{\xi+2}\}$ since the set $\mathscr{B}_0$ of functional elements corresponding to the t-operation $Tf_{\xi+2}$ is $\{L, R, Id_{\xi+1}, Ml_{\xi+1}\}$.

Proposition 14.20* may be extended to the following two Hierarchy tf-Conservativeness Theorems.

Proposition 15.9*. Let $\varphi \in \mathscr{F}_{\xi+2}$, $\mathscr{B} \subseteq \mathscr{F}_{\xi+2}$ and φ be recursive in $\mathscr{B} \cup \{Id_\eta, Ml_\eta / \eta > \xi\} \cup \{Tf_{\eta+2} / \eta \geq \xi\}$. Then φ is $tf_{\xi+2}$-recursive in $\mathscr{B}$.

Proof. Writing $\mathscr{B}_\eta$, $\mathscr{B}_{<\eta}$ respectively for

$$\mathscr{B} \cup \{Id_\zeta, Ml_\zeta / \xi < \zeta \leq \eta\} \cup \{Tf_{\zeta+2} / \xi + 2 \leq \zeta + 2 \leq \eta\},$$
$$\mathscr{B} \cup \{Id_\zeta, Ml_\zeta / \xi < \zeta < \eta\} \cup \{Tf_{\zeta+2} / \xi + 2 \leq \zeta + 2 < \eta\},$$

φ is recursive in $\mathscr{B}_{\eta_0}$ for a certain η_0. Assuming that $\eta_0 > \xi + 2$, there are two possible cases.

If η_0 is not of the form $\zeta + 2$, then 12.38* implies that φ is a mapping over $\mathscr{F}_{\eta_0}$ recursive in $\mathscr{B}_{<\eta_0}$, hence $\varphi = \varphi(I)_{\eta_0}$ is an element recursive in $\mathscr{B}_{<\eta_0}$.

If $\eta_0 = \zeta + 2$, then an application of 14.20* to the consecutive spaces $\mathscr{S}_{\eta_0}$, $\mathscr{S}_{\eta_0+1}$ implies that φ is a mapping over $\mathscr{F}_{\eta_0}$ tf_{η_0}-recursive in $\mathscr{B}_{<\eta_0}$; hence $\varphi = \varphi(I)_{\eta_0}$ is an element tf_{η_0}-recursive in $\mathscr{B}_{<\eta_0}$. The implication (3)$\Rightarrow$(2) of 14.20* implies that φ is recursive in $\mathscr{B}_{<\eta_0}$ in this case as well.

Therefore, there is an $\eta_1 < \eta_0$ such that φ is recursive in $\mathscr{B}_{\eta_1}$. We get in finitely many steps an $\eta_n \leq \xi + 2$ such that φ is recursive in $\mathscr{B}_{\eta_n}$; hence φ is recursive in

$$\mathscr{B} \cup \{Id_{\xi+1}, Ml_{\xi+1}, Id_{\xi+2}, Ml_{\xi+2}, Tf_{\xi+2}\}.$$

The implication (2)$\Rightarrow$(1) of 14.20* ensures that φ is a mapping over $\mathscr{F}_{\xi+2}$ $tf_{\xi+2}$-recursive in $\mathscr{B}$, hence $\varphi = \varphi(I)_{\xi+2}$ is an element $tf_{\xi+2}$-recursive in $\mathscr{B}$. The proof is complete.

Proposition 15.10*. Let ξ be a limit ordinal or a successor to a limit ordinal, $\varphi \in \mathscr{F}_\xi$, $\mathscr{B} \subseteq \mathscr{F}_\xi$ and φ be recursive in

$$\mathscr{B} \cup \{Id_\eta, Ml_\eta / \eta \geq \xi\} \cup \{Tf_{\eta+2} / \eta + 2 > \xi\}.$$

Then φ is recursive in $\mathscr{B}$.

Proof. Writing $\mathscr{B}_\eta$ for

$$\mathscr{B} \cup \{Id_\zeta, Ml_\zeta / \xi \leq \zeta \leq \eta\} \cup \{Tf_{\zeta+2} / \xi < \zeta + 2 \leq \eta\}$$

and following the proof of 15.9*, we get an $\eta_n \leq \xi$ such that φ is recursive in $\mathscr{B}_{\eta_n} \subseteq \mathscr{B} \cup \{Id_\xi, Ml_\xi\}$; hence φ is recursive in $\mathscr{B}$ by 12.38*. This completes the proof.

There are two statements to extend 14.20* since we have no transfer operations over $\mathscr{F}_\xi, \mathscr{F}_{\xi+1}$ for limit ξ. The corresponding two Hierarchy First tf-Recursion Theorems below seem closer to the original theorem of Platek.

Proposition 15.11*. Let $\varphi \in \mathscr{F}_{\xi+3}$ and $\mathscr{B} \subseteq \mathscr{F}_{\xi+2}$. Then the following are equivalent.

(1) φ is (prime) recursive in

$$\mathscr{B} \cup \{Id_{\xi+1}, Ml_{\xi+1}, Id_{\xi+2}, Ml_{\xi+2}, Tf_{\xi+2}\}.$$

(2) $\varphi \in cl(\mathscr{B} \cup \{L, A, Id_{\xi+1}, Ml_{\xi+1}, Id_{\xi+2}, Ml_{\xi+2}, Tr_{\xi+2}, It_{\xi+2}, Tf_{\xi+2}\}/\circ)$.

(3) $\varphi \in cl(\mathscr{B} \cup \{L, A\} \cup \{Id_\eta, Ml_\eta/\eta > \xi\}$
$\cup \{Tr_\eta, It_\eta/\eta > \xi + 1\} \cup \{Tf_{\eta+2}/\eta \geq \xi\}/\circ)$.

If $\varphi \in \mathscr{F}_{\xi+2}$, then (1) can be replaced by 'φ is $tf_{\xi+2}$-recursive in $\mathscr{B}$'. Exercise 14.3** allows us to drop $Tr_{\xi+2}$ in (2).

Proof. The first two assertions are equivalent by 14.20*. The implication (1)$\Rightarrow$(3) is immediate. Assuming (3), we obtain that φ is $tf_{\xi+3}$-recursive in $\mathscr{B} \cup \{Id_{\xi+1}, Ml_{\xi+1}, Tf_{\xi+2}\}$ by 15.9*, which gives (1) by the implication (3)$\Rightarrow$(2) of 14.20*.

If $\varphi \in \mathscr{F}_{\xi+2}$, then (1) implies that φ is $tf_{\xi+2}$-recursive in $\mathscr{B}$ by 14.20*. The proof is complete.

Proposition 15.12*. Let $\varphi \in \mathscr{F}_{\xi+1}$, $\mathscr{B} \subseteq \mathscr{F}_\xi$ and suppose that ξ is not of the form $\xi_1 + 2$. Then the following are equivalent.

(1) φ is (prime) recursive in $\mathscr{B} \cup \{Id_\xi, Ml_\xi\}$.

(2) $\varphi \in cl(\mathscr{B} \cup \{L, A, Id_\xi, Ml_\xi, Tr_\xi, It_\xi\}/\circ)$.

(3) $\varphi \in cl(\mathscr{B} \cup \{L, A\} \cup \{Id_\eta, Ml_\eta, Tr_\eta, It_\eta/\eta \geq \xi\} \cup \{Tf_{\eta+2}/\eta + 2 > \xi\}/\circ)$.

If $\varphi \in \mathscr{F}_\xi$, then (1) can be replaced by 'φ is recursive in $\mathscr{B}$'.

Proof. The equivalence (1)$\Leftrightarrow$(2) was established in 15.8*, while (1)$\Rightarrow$(3) is immediate. Finally, (3) implies (1) by 15.10*.

If $\varphi \in \mathscr{F}_\xi$, then (1) implies that φ is recursive in $\mathscr{B}$ by 12.38*. The proof is complete.

Some concluding remarks.

The carrier of a monotonic hierarchy with transfers seems to form a model of a typed $\lambda\mu$-calculus with combinators I, L, R, Ml_0 and $\Pi_0 = \Pi \restriction \mathscr{F}_0^2$; the μ-operation concerned is introduced in the exercises below. Instead of $\mathscr{M}\mathscr{H}$ one may consider initial segments $\mathscr{F}_\xi, \xi$ a limit ordinal.

Transfers in a hierarchy are most probably independent, i.e. $Tf_{\xi+2}$ is not recursive in $\{Id_\eta, Ml_\eta/\eta \geq 0\} \cup \{Tf_{\eta+2}/\eta \neq \xi\}$ for all ξ. (Though we have not even proved that $Tf_{\xi+2}$ is not recursive in $\{Id_\eta, Ml_\eta/\eta \geq 0\}$.) Therefore, the transfer operation is necessary for the abovementioned combinatorial completeness.

It may be of some interest to study recursion in certain specific elements, e.g. elements embodying quantifiers. Among the possible candidates are the mappings Q_σ, $Q_{\sigma,\mathscr{C}}$ of chapter 13.

The work of Platek referred to above studies what can be described in IOS-terms as the ω-segment of a monotonic hierarchy based on example 4.7. Therefore, the considerations of this chapter embody Platek's work in Generalized Recursion Theory in a more general setting.

EXERCISES TO CHAPTER 15

Exercise 15.1. Let $\varphi, \psi \in \mathscr{F}_\xi$ and $\varphi, \psi \leq \chi$. Show that there is a $\chi_1 \in \mathscr{F}_\xi$ such that $\varphi, \psi \leq \chi_1$.

Hint. Use transfinite induction on $r(\chi)$.

Exercise 15.2. Show that $\varphi \in \mathscr{F}_\xi$ iff $\varphi = Ml_\xi(\varphi Id_\xi, I)$. Therefore, if $\varphi \in \mathscr{F}_\xi$, then φ is polynomial in φId_ξ, Ml_ξ.

Hint. $\varphi \in \mathscr{F}_\xi$ iff $\varphi = \varphi(I)_\xi$.

Exercise 15.3. Show that whenever $\xi < \eta$, then

$$Tr_\xi = RMl_\eta(It_\eta(Ml_\xi(LId_\eta, LId_\xi), Ml_\xi(RId_\eta, \mathrm{RId}_\xi)), I).$$

Hint. Observing that $\mathscr{S}_{\xi+1}$ is a subspace of $\mathscr{S}_\eta$, get $Tr_\xi = RIt_\eta((Ml_\xi(L, LId_\xi), Ml_\xi(R, RId_\xi)))_\eta$ by the proof of 12.28, then use 15.4.

Exercise 15.4. Show that if $\xi < \eta$, then

$$It_\xi = RMl_\eta(It_\eta(Id_\xi, Ml_\xi Id_\eta), I).$$

Hint. As for the previous exercise, with 12.28 replaced by 12.29.

Write $\mu_\xi\theta.$ ——— for the least θ in $\mathscr{F}_\xi$ such that——— $\leq \theta$, provided it exists.

Exercise 15.5. Let $\varphi \in \mathscr{F}_{\xi+1}$. Show that $\mu_\xi\theta.\varphi(\theta)_\xi$ exists and equals $Ml_\xi(R[\varphi R], I)$. Show that $\mu_\eta\theta.\varphi(\theta)_\eta = R[\varphi R]$, provided $\xi < \eta$.

Hint. 12.27 implies that $\theta_0 = \mu_\xi\theta.\varphi(\theta)_\xi$ exists and $\theta_0 Id_\xi = R[\varphi R]$. Make use of exercise 15.2.

Exercise 15.6. Prove that there is no collection operation over $\mathscr{F}_\xi$, $\xi \geq \omega$.

Hint. Suppose that $Co: \mathscr{F}_\xi^\omega \to \mathscr{F}_\xi$ satisfies the axiom cA_3 of chapter 11. Take $\varphi = Co\{\varphi_n\}$, where $r(\varphi_n) = n$ for all n. Construct a sequence $\xi_1 > \cdots > \xi_n \geq \omega$ such that $\varphi(I)_{\xi_1} \ldots (I)_{\xi_i} \in \mathscr{F}_{\xi_{i+1}+1}$ for all $1 \leq i < n$, and $\psi = \varphi(I)_{\xi_1} \ldots (I)_{\xi_n} \in \mathscr{F}_\omega$. Assuming $r(\psi) = m$, get $\varphi_{m+1} = \varphi_{m+1}(I)_{\xi_1} \ldots (I)_{\xi_n} = m + 1\psi \in \mathscr{F}_m$, which is not the case.